MATHEMATICAL PHYSICS

for

M.Sc. Entrance

IIT-JAM | JNU | TIFR | JEST | BHU | ISM | JNCASR | DU

First Edition : April 2018

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Chapter 1

VECTOR ALGEBRA

1.1 Basic Review of Vectors

Definition:

Physical quantities having both magnitude and a definite direction in space. It should follow the law of vector addition.

Example: Velocity, Acceleration, Momentum, Force, Electric Field, Torque, etc.

Note: Current is a physical quantity that has both magnitude and direction but it does not follow the law of vector addition. So, current is a scalar quantity.

Various type of vectors:

(1) Equal vectors: Vectors having same magnitude and same direction.

(2) Null Vectors: Vectors having coincident initial and terminal point i.e. its magnitude is zero and it has any arbitrary direction.

(3) Unit Vector: Vector having unit magnitude. Unit vector along $\vec{a}$ is $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

(4) Reciprocal Vector: Vector having same direction as $\vec{a}$ but magnitude reciprocal to that of $\vec{a}$, is known as the reciprocal vector of $\vec{a}$. Reciprocal vector of $\vec{a}$ is $\vec{a}^{-1} = \frac{1}{|\vec{a}|}\hat{a}$

(5) Negative Vector: Vectors having same magnitude as $\vec{a}$ but direction opposite to that of $\vec{a}$, is known as the negative vector of $\vec{a}$. Negative vector as $\vec{a}$ is $-\vec{a} = -|a|\hat{a}$

Orthogonal Resolution of Vectors:

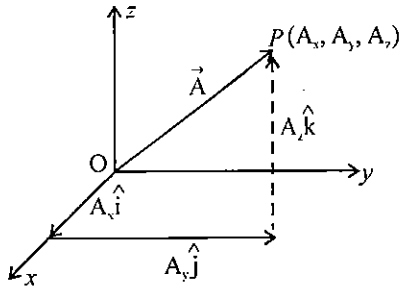
Any vector $\vec{A}$ in the 3-D right- handed rectangular cartesian coordinate system can be represented as

$$\overrightarrow{OP} = \vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k},$$

where, $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors in direction of x, y and z axis respectively and A_x, A_y, A_z are the rectangular components of vector $\vec{A}$ along x, y, z axis.

Magnitude of vector $\vec{A}$ is $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Unit vector along $\vec{A}$ is $\hat{A} = \vec{A} / |\vec{A}| = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right) / \sqrt{A_x^2 + A_y^2 + A_z^2}$



Direction cosines of vector $\vec{A}$:

If $\vec{A}$ makes angles α, β, γ with x, y and z axes respectively, then direction cosines of $\vec{A}$ are defined as

$$l = \cos\alpha = \frac{A_x}{A};\ m = \cos\beta = \frac{A_y}{A};\ n = \cos\gamma = \frac{A_z}{A} \quad \text{and} \quad l^2 + m^2 + n^2 = 1$$

Note: Unit vector along $\vec{A}$ can be written as $\hat{A} = l\hat{i} + m\hat{j} + n\hat{k}$

1.2 Products of Vectors

Scalar Product or Dot Product:

Dot product of two vectors $\vec{a}$ and $\vec{b}$ are defined as

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta \Rightarrow \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

Note:

(i) Dot product is commutative in nature i.e. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

(ii) For two mutually perpendicular vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\cdot\vec{b} = 0$

(iii) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot i = 0,\ \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

(iv) If $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, then $\vec{a}.\vec{b} = a_xb_x + a_yb_y + a_zb_z$

(v) Projection of $\vec{A}$ on $\vec{B} = \vec{A}.\hat{B}$

Vector Product or Cross Product :

Cross product of two vectors $\vec{a}$ and $\vec{b}$ are defined as $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n}$

where $\hat{n}$ is unit vector normal to the plane containing $\vec{a}$ and $\vec{b}$.

Note:

(i) Cross product is not commutative i.e. $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

(ii) For two collinear vectors (parallel or anti-parallel vectors) $\vec{a}\times\vec{b} = 0$.

(iii) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0,\ \hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$

(iv) If $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

Example 1: The range of x for which the angle between the vectors

$$\vec{A} = 2x^2\hat{i} + 4x\hat{j} + \hat{k} \text{ and } \vec{B} = 7\hat{i} - 2\hat{j} + x\hat{k}$$

is obtuse, is equal to

(a) $x < 0$ (b) $x > 1/2$ (c) $0 < x < 1/2$ (d) None of these

Soln: Since, $90^0 < \theta < 180^0$, then $\cos\theta < 0 \Rightarrow \vec{A}.\vec{B} < 0 \Rightarrow 14x^2 - 7x < 0 \Rightarrow x(2x-1) < 0$

Case I: $x > 0$ and $(2x-1) < 0 \Rightarrow x < \frac{1}{2}$. Therefore, $0 < x < \frac{1}{2}$

Case II: $x < 0$ and $(2x-1) > 0 \Rightarrow x > \frac{1}{2}$. Both of them cannot be satisfied simultaneously.

Correct option is (c)

Example 2: Given: $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ are the sides of

(a) an isosceles triangle (b) a right-angled triangle

(c) an equilateral triangle (d) none of these

Soln: For the given set of vectors, $\vec{a}.\vec{c} = 0 \Rightarrow \vec{a}$ is perpendicular to $\vec{c}$ and

$$|\vec{a}|^2 + |\vec{c}|^2 = 14 + 21 = 35 = |\vec{b}|^2$$

Therefore, $\vec{a}$, $\vec{c}$ will correspond to base and height respectively and $\vec{b}$ will correspond to hypotenus i.e. the given set of vectors are sides of a right-angled triangle.

Correct option is (b)

Example 3: Given : $\vec{P} = 2\hat{i} - 3\hat{j}, \vec{Q} = \hat{i} + \hat{j} - \hat{k}, \vec{R} = 3\hat{i} - \hat{k}$, the vector $\vec{V}$, which is orthogonal to $\vec{P}$ and $\vec{Q}$ and having unit scalar product with $\vec{R}$, can be estimated to be

(a) $(3\hat{i} + 2\hat{j} + 5\hat{k})/4$ (b) $(3\hat{i} + 2\hat{j} - 5\hat{k})/4$ (c) $(3\hat{i} - 2\hat{j} + 5\hat{k})/4$ (d) $(3\hat{i} - 2\hat{j} - 5\hat{k})/4$

Soln: Unit vector along $\vec{V}$ is $\hat{V} = \frac{\vec{P} \times \vec{Q}}{|\vec{P} \times \vec{Q}|} = \frac{3\hat{i} + 2\hat{j} + 5\hat{k}}{\sqrt{38}}$

According to the question, $\vec{V}.\vec{R} = 1 \Rightarrow (V\hat{V}).\vec{R} = 1 \Rightarrow V = \frac{\sqrt{38}}{4}$

Correct option is (a)

Example 4: For any vector $\vec{a}$, the value of $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2$ is equal to

(a) $|\vec{a}|^2$ (b) $2|\vec{a}|^2$ (c) $\vec{a}^2$ (d) $2\vec{a}^2$

Soln: Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ (Here, a_1, a_2, a_3 are not constants)

$\therefore \quad \vec{a}\times i = a_2\hat{k} - a_3\hat{j}$

$\left|\vec{a}\times\hat{i}\right| = \sqrt{a_2^2 + a_3^2}$

Similarly, $\left|\vec{a}\times\hat{j}\right| = \sqrt{a_1^2 + a_3^2}$ and $\left|\vec{a}\times\hat{k}\right| = \sqrt{a_1^2 + a_2^2}$

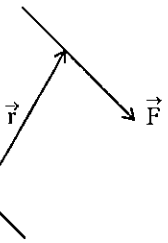
$\therefore \quad \left|\vec{a}\times\hat{i}\right|^2 + \left|\vec{a}\times\hat{j}\right|^2 + \left|\vec{a}\times\hat{k}\right|^2 = 2\left(a_1^2 + a_2^2 + a_3^2\right) = 2\left|\vec{a}\right|^2$

Correct option is (b)

Example 5: If $\vec{F} = \hat{i} + 3\hat{j} - \hat{k}$ acts at the point (1, 1, 1), then the torque of $\vec{F}$ about the line $\vec{r} = 3\hat{i} + 2\hat{k} + (2\hat{i} - 2\hat{j} + \hat{k})t$ is_____.

Soln: We first find the vector torque about a point on the line, say the point (3, 0, 2).

This is $\vec{r}\times\vec{F}$, where $\vec{r}$ is the vector from the point about which we want the torque, to the point at which $\vec{F}$ acts, that is, from (3, 0, 2) to (1, 1, 1).



$\therefore \quad \vec{r} = \left(\hat{i} + \hat{j} + \hat{k}\right) - \left(3\hat{i} + 2\hat{k}\right) = -2\hat{i} + \hat{j} - \hat{k}$

$$\therefore \quad \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -1 \\ 1 & 3 & -1 \end{vmatrix} = 2\hat{i} + 3\hat{j} - 7\hat{k}$$

The torque about the line is $\hat{n}\cdot\left(\vec{r}\times\vec{F}\right)$ where $\hat{n}$ is a unit vector along the line $\hat{n} = \frac{1}{3}\left(2\hat{i} - 2\hat{j} + \hat{k}\right)$

Therefore, torque about the line, $\hat{n}\cdot\left(\vec{r}\times\vec{F}\right) = 1$

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Correct answer is (1)

Example 6: Which of the following vectors is orthogonal to the vector $\left(a\hat{i} + b\hat{j}\right)$, where a and b $\left(a \neq b\right)$ are constants, and $\hat{i}$ and $\hat{j}$ are unit orthogonal vectors?

(a) $-b\hat{i} + a\hat{j}$ (b) $-a\hat{i} + b\hat{j}$ (c) $-a\hat{i} - b\hat{j}$ (d) $-b\hat{i} - a\hat{j}$

Soln: $\left(a\hat{i} + b\hat{j}\right).\left(-b\hat{i} + a\hat{j}\right) = -ab + ab = 0$

Correct option is (a)

Scalar Triple Product:

Scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are defined as

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=[abc]$$

Note:

(i) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$ i.e. $[abc] = [bca] = [cab]$

(ii) Volume of a parallelopiped having $\vec{a},\vec{b},\vec{c}$ as concurrent edges is :V$=\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$

(iii) If $\vec{a},\vec{b},\vec{c}$ are coplanar vectors, then, $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = 0$

Vector Triple Product:

Vector triple product of three vectors $\vec{a},\vec{b}$ and $\vec{c}$ are defined as

$$\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$$

It represents a vector coplanar with $\vec{b}$ and $\vec{c}$ and perpendicular to $\vec{a}$.

Note:

(i) $\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=0$

Example 7: The value of m for which the vectors $\vec{A}=\hat{i}-\hat{j}-2\hat{k}, \vec{B}=3\hat{i}+5\hat{j}+6\hat{k}, \vec{C}=-\hat{i}+4\hat{j}+m\hat{k}$ will be co-planar is

(a) 13/2 (b) 7/2 (c) 5/2 (d) 11/2

Soln: Since, $\vec{A},\vec{B},\vec{C}$ are co-planar, $\vec{A}.\left(\vec{B}\times\vec{C}\right)=0\Rightarrow\begin{vmatrix} 1 & -1 & -2 \\ 3 & 5 & 6 \\ -1 & 4 & m \end{vmatrix}=0\Rightarrow m=\frac{13}{2}$

Correct option is (a)

Example 8: If $\vec{b}=\hat{i}\times\left(\vec{a}\times\hat{i}\right)+\hat{j}\times\left(\vec{a}\times\hat{j}\right)+\hat{k}\times\left(\vec{a}\times\hat{k}\right)$, then $\vec{b}$ can be simplified to

(a) 0 (b) $\vec{a}$ (c) $2\vec{a}$ (d) None of these

Soln: $\vec{b}=\hat{i}\times\left(\vec{a}\times\hat{i}\right)+\hat{j}\times\left(\vec{a}\times\hat{j}\right)+\hat{k}\times\left(\vec{a}\times\hat{k}\right)$

$=\vec{a}\left(\hat{i}.\hat{i}\right)-\hat{i}\left(\hat{i}.\vec{a}\right)+\vec{a}\left(\hat{j}.\hat{j}\right)-\hat{j}\left(\hat{j}.\vec{a}\right)+\vec{a}\left(\hat{k}.\hat{k}\right)-\hat{k}\left(\hat{k}.\vec{a}\right)=3\vec{a}-\vec{a}=2\vec{a}$

$=\left(\vec{a}+\vec{a}+\vec{a}\right)-\hat{i}\left(a_x\right)-\hat{j}\left(a_y\right)-\hat{k}\left(a_z\right)$

$=3\vec{a}-\vec{a}-=2\vec{a}$

Correct option is (c)

Example 9: Three unit vectors $\vec{a},\vec{b},\vec{c}$ ($\vec{b}$ and $\vec{c}$ are not parallel) are such that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\frac{\sqrt{3}}{2}\vec{c}$. The angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, respectively are

(a) $30^0,90^0$ (b) $150^0,90^0$ (c) $60^0,90^0$ (d) $90^0,30^0$

Soln: $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\frac{\sqrt{3}}{2}\vec{c}\Rightarrow\vec{b}\left(\vec{a}.\vec{c}\right)-\vec{c}\left(\vec{a}.\vec{b}\right)=\frac{\sqrt{3}}{2}\vec{c}$

Comparing coefficients of $\vec{b}$ and $\vec{c}$ on both sides.

$\vec{a}\cdot\vec{c}=0\ \Rightarrow\ \vec{a}\perp\vec{c}$

$\vec{a}\cdot\vec{b}=\frac{\sqrt{3}}{2}\Rightarrow ab\cos\theta=-\frac{\sqrt{3}}{2}\Rightarrow\cos\theta=-\frac{\sqrt{3}}{2}\Rightarrow\theta=150°$

The angle which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$ are $150^0,90^0$ respectively.

Correct option is (b)

Example 10: If $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{\pi}{3}$ (d) $\frac{3\pi}{4}$

Soln: Since, $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$ or $\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})=\frac{\vec{b}}{\sqrt{2}}+\frac{\vec{c}}{\sqrt{2}}$

On comparing both side, we have $-(\vec{a}\cdot\vec{b})=\frac{1}{\sqrt{2}}\Rightarrow\vec{a}\cdot\vec{b}=-\frac{1}{\sqrt{2}}$

If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\vec{a}\cdot\vec{b}=\frac{1}{\sqrt{2}}$

Therefore, $\theta=\frac{3\pi}{4}$.

Correct option is (d)

1.3 Gradient, Divergence and Curl

Gradient of a Scalar Field:

Gradient of a continuously differentiable scalar function $\phi(x, y, z)$ is mathematically defined as:

$$grad\ \phi=\vec{\nabla}\phi=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\phi=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$$

where, $\vec{\nabla}=\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\Rightarrow$ 'del' or 'grad' or 'nabla' operator

Physical interpretation: Gradient of scalar function ϕ at any point P(x, y, z) is a vector quantity, whose magnitude is equal to the rate of change of scalar function ϕ with distance along the normal to level surface and its direction is along the normal to the level surface at that point. The direction of $\vec{\nabla}\phi$ is that along which the rate of change of ϕ is maximum.

Level surface: At each point of the level surafce, the value of scalar function ϕ will be same.

Example: Equipotential surface on which value of electrostatic potential is same at all points

Note:

(i) Normal vector to the level surface: $\vec{\nabla}\phi$

(ii) Unit normal vector to the level surface: $\hat{n} = \dfrac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|}$

Directional Derivative: Directional derivative of ϕ in the direction of $\vec{A}$ is defined as rate of change of ϕ with distance along the direction of $\vec{A}$. It is mathematically defined as the component of $\vec{\nabla}\phi$ in the direction of vector $\vec{A}$ i.e. $\vec{\nabla}\phi \cdot \hat{A} = \vec{\nabla}\phi \cdot \dfrac{\vec{A}}{\left|\vec{A}\right|}$

Tangent Plane and Normal to the level surface:

Consider $\phi(x, y, z) = c$ be the equation of a level surface, and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point P (x,y,z) on this surface.

Tangent plane at P: $\vec{\nabla}\phi$ is a vector normal to the surface i.e. it is perpendicular to the tangent plane at P. Let, $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$ be the position vector of any point on the tangent plane at P to the surface. Therefore, $\vec{R} - \vec{r} = (X - x)\hat{i} + (Y - y)\hat{j} + (Z - z)\hat{k}$ lies in the tangent plane at P and it will be perpendicular to $\vec{\nabla}\phi$

i.e. $\left(\vec{R} - \vec{r}\right).\vec{\nabla}\phi = 0$

$\Rightarrow (X - x)\dfrac{\partial\phi}{\partial x} + (Y - y)\dfrac{\partial\phi}{\partial y} + (Z - z)\dfrac{\partial\phi}{\partial z} = 0$, which is the tangent plane at point P.

Example 11: Find the unit vector perpendicular to the surface $x^2 + y^2 + z^2 = 6$ at point P (2,1,1).

Soln: Unit normal to the surface $\hat{n} = \left.\dfrac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|}\right|_{(2,1,1)} = \left.\dfrac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}\right|_{(2,1,1)} = \dfrac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$

Example 12: Find the angle between the two surfaces $5x + y + z = 1$ and $3x + 3y + 3z = 5$.

Soln: Given, $\phi_1 : 5x + y + z = 1$ and $\phi_2 : 3x + 3y + 3z = 5$

Therefore, $\cos\theta = \dfrac{\vec{\nabla}\phi_1.\vec{\nabla}\phi_2}{\left|\vec{\nabla}\phi_1\right|\left|\vec{\nabla}\phi_2\right|} = \dfrac{\left(5\hat{i} + \hat{j} + \hat{k}\right).\left(3\hat{i} + 3\hat{j} + 3\hat{k}\right)}{\sqrt{27}.\sqrt{27}} = \dfrac{21}{27} \Rightarrow \theta = \cos^{-1}\left(\dfrac{7}{9}\right)$

Example 13: Find the directional derivative of $f(x,y)=x^2y^3-y^4$ at (2,1) in the direction given by the angle $\theta=\frac{\pi}{4}$.

Soln: Unit vector along which the directioanl derivative is to be calculated is

$$\hat{A}=\cos\frac{\pi}{4}\hat{i}+\sin\frac{\pi}{4}\hat{j}=\frac{1}{\sqrt{2}}\left(\hat{i}+\hat{j}\right)$$

Therefore, directional derivative will be

$$\vec{\nabla}f\Big|_{(2,1)}.\hat{A}=\left[\left(2xy^3\right)\hat{i}+\left(3x^2y^2-4y^3\right)\hat{j}\right]_{(2,1)}.\frac{1}{\sqrt{2}}\left(\hat{i}+\hat{j}\right)=6\sqrt{2}$$

Example 14: If grad $\phi=\frac{\hat{r}}{r^4}$ & $\phi(r)=0$ at $r=1$, then find ϕ.

Soln: $\vec{\nabla}\phi=\frac{\hat{r}}{r^4}\Rightarrow\frac{\partial\phi}{\partial r}\hat{r}=\frac{\hat{r}}{r^4}\Rightarrow\frac{\partial\phi}{\partial r}=\frac{1}{r^4}\Rightarrow\phi=-\frac{1}{3r^3}+C$

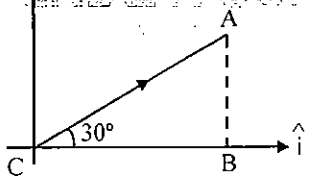
Since, at $r=1$, $\phi(r)=0\Rightarrow 0=-\frac{1}{3}+C\Rightarrow C=\frac{1}{3}$

Putting this we get $\phi(r)=-\frac{1}{3r^3}+\frac{1}{3}$

Example 15: For the function $\phi(x,y)=\frac{x}{x^2+y^2}$, find the magnitude of the directional derivative along a line making an angle 30° with the positive x-axis at (0, 2)

Soln: $\vec{\nabla}\phi=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\frac{x}{x^2+y^2}=\hat{i}\left(\frac{1}{x^2+y^2}-\frac{x(2x)}{\left(x^2+y^2\right)^2}\right)-\hat{j}\frac{x(2y)}{\left(x^2+y^2\right)^2}$

$$=\hat{i}\frac{y^2-x^2}{\left(x^2+y^2\right)^2}-\hat{j}\frac{2xy}{\left(x^2+y^2\right)^2}$$



$\vec{\nabla}\phi$ at the point (0, 2)

$$=\hat{i}\frac{4-0}{(0+4)^2}-\hat{j}\frac{2(0)(2)}{(0+4)^2}=\frac{\hat{i}}{4}$$

Direction along which Directional derivative is to be calculated is

$$\overline{CA}=\cos 30^\circ\hat{i}+\sin 30^\circ\hat{j}=\frac{\sqrt{3}}{2}\hat{i}+\frac{1}{2}\hat{j}$$

Directional derivative at the point (0, 2) in the direction $\overline{CA}$

$$=\frac{\hat{i}}{4}.\left(\frac{\sqrt{3}}{2}\hat{i}+\frac{1}{2}\hat{j}\right)=\frac{\sqrt{3}}{8}$$

Example 16: Find the directional derivative of $\vec{V}^2$, where $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$, at the point (2, 0, 3) in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point (3, 2, 1).

Soln: $V^2 = \vec{V}.\vec{V} = V^2 = \vec{V}.\vec{V} = \left(xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}\right).\left(xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}\right) = x^2y^4 + z^2y^4 + x^2z^4$

$$\vec{\nabla}V^2 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2y^4 + z^2y^4 + x^2z^4\right)$$

$$= \left(2xy^4 + 2xz^4\right)\hat{i} + \left(4x^2y^3 + 4y^3z^2\right)\hat{j} + \left(2y^4z + 4x^2z^3\right)\hat{k}$$

$$\vec{\nabla}V^2 \text{ at } (2, 0, 3) = \left(0 + 2\times2\times81\right)\hat{i} + \left(0 + 0\right)\hat{j} + \left(0 + 4\times4\times27\right)\hat{k}$$

$$= 324\hat{i} + 432\hat{k} = 108\left(3\hat{i} + 4\hat{k}\right)$$

$$\text{Normal to } x^2 + y^2 + z^2 - 14 = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2 - 14\right)$$

$$= \left(2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right)$$

Normal vector at $(3, 2, 1) = 6\hat{i} + 4\hat{j} + 2\hat{k}$

$$\text{Unit normal vector} = \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{36 + 16 + 4}} = \frac{2\left(3\hat{i} + 2\hat{j} + \hat{k}\right)}{2\sqrt{14}} = \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$$

$$\text{Direction derivative along the normal} = 108\left(3\hat{i} + 4\hat{k}\right).\frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}} = \frac{108\times(9 + 4)}{\sqrt{14}} = \frac{1404}{\sqrt{14}}$$

Example 17: The plane tangent to $x^2 + y^2 + z^2 = 4$ at the point $\left(1, 1, \sqrt{2}\right)$ is

(a) $z = \frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1) + \sqrt{2}$ (b) $z = -\frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1) + \sqrt{2}$

(c) $z = -(x-1) - (y-1) + \sqrt{2}$ (d) $z = -\frac{1}{\sqrt{2}}(x-1) + (y-1) + \sqrt{2}$

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Soln: Let $\phi : x^2 + y^2 + z^2 - 4 = 0$

$$\therefore \quad \frac{\partial\phi}{\partial x} = 2x \Rightarrow \left.\frac{\partial\phi}{\partial x}\right|_{1,1,\sqrt{2}} = 2, \quad \frac{\partial\phi}{\partial y} = 2y \Rightarrow \left.\frac{\partial\phi}{\partial y}\right|_{1,1,\sqrt{2}} = 2, \quad \frac{\partial\phi}{\partial z} = 2z \Rightarrow \left.\frac{\partial\phi}{\partial y}\right|_{1,1,\sqrt{2}} = 2\sqrt{2}$$

Therefore, equation of tangent plane at the point $\left(1, 1, \sqrt{2}\right)$

$$(x-1)\times2 + (y-1)\times2 + \left(z - \sqrt{2}\right)\times2\sqrt{2} = 0$$

$$\Rightarrow z - \sqrt{2} = -\frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1)$$

Correct option is (b)

Example 18: Suppose the temperature at (x, y, z) is given by $T = xyz$. In what direction can you go from the point $(1,1,1)$ to maintain the same temperature?

(a) $-\hat{i} + 2\hat{j} - \hat{k}$ (b) $\hat{i} + \hat{j} + \hat{k}$ (c) $\hat{i} - \hat{j} + \hat{k}$ (d) none

Soln: To maintain the same temperature, we must travel perpendicular to the direction of $grad\,(T)$.

Here, $\vec{\nabla}T = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$\therefore \quad \vec{\nabla}T\big|_{(1,1,1)} = \hat{i} + \hat{j} + \hat{k}$

and $\left(\hat{i} + \hat{j} + \hat{k}\right) \cdot \left(-\hat{i} + 2\hat{k} - \hat{k}\right) = 0$

Correct option is (a)

Example 19: If 'f' is a scalar function such that $\vec{\nabla}f = (3 + 2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$, then the value of f at the point (1, 1) is_____.

Soln: Since, $\vec{\nabla}f = (3+2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$

$$\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = (3+2xy)\hat{i} + \left(x^2 - 3y^2\right)\hat{j}$$

$$\therefore \quad \frac{\partial f}{\partial x} = 3 + 2xy \Rightarrow f = 3x + x^2y \qquad \text{... (i)}$$

and $$\frac{\partial f}{\partial y} = x^2 - 3y^2 \Rightarrow f = x^2y - y^3 \qquad \text{... (ii)}$$

Hence, $f(x, y) = x^2y - y^3 + 3x$

Therefore, value of $f(x, y)$ at (1, 1) is 3.

Correct answer is (3)

Example 20: The unit vector normal to the surface $3x^2 + 4y = z$ at the ponit (1, 1, 7) is:

(a) $\left(-6\hat{i} + 4\hat{j} + \hat{k}\right)/\sqrt{53}$ (b) $\left(4\hat{i} + 6\hat{j} - \hat{k}\right)/\sqrt{53}$

(c) $\left(6\hat{i} + 4\hat{j} - \hat{k}\right)/\sqrt{53}$ (d) $\left(4\hat{i} + 6\hat{j} + \hat{k}\right)/\sqrt{53}$

Soln: Unit vector normal to the surface $3x^2 + 4y = z$ at the ponit (1, 1, 7) is

$$\hat{n} = \frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|} = \frac{\left(6x\,\hat{i} + 4\hat{j} - \hat{k}\right)}{\sqrt{36x^2 + 16 + 1}}\Bigg|_{(1,1,7)} = \frac{\left(6\,\hat{i} + 4\hat{j} - \hat{k}\right)}{\sqrt{53}}$$

Correct option is (c)

Example 21: For the function $\phi = x^2y + xy$ the value of $\left|\vec{\nabla}\phi\right|$ at $x = y = 1$ is

(a) 5 (b) $\sqrt{5}$ (c) 13 (d) $\sqrt{13}$

Soln: $\phi = x^2y + xy \Rightarrow \vec{\nabla}\phi = \left(2xy + y\right)\hat{i} + \left(x^2 + x\right)\hat{j}$

$\vec{\nabla}\phi\big|_{(1,1,1)} = 3\hat{i} + 2\hat{j}$; $\left|\vec{\nabla}\phi\right|_{(1,1,1)} = \sqrt{13}$

Correct option is (d)

Example 22: The direction of $\vec{\nabla}f$ for a scalar field $f(x,y,z)=\frac{1}{2}x^2-xy+\frac{1}{2}z^2$ at the point $P(1, 1, 2)$ is

(a) $\frac{\left(-\hat{j}-2\hat{k}\right)}{\sqrt{5}}$ (b) $\frac{\left(-\hat{j}+2\hat{k}\right)}{\sqrt{5}}$ (c) $\frac{\left(\hat{j}-2\hat{k}\right)}{\sqrt{5}}$ (d) $\frac{\left(\hat{j}+2\hat{k}\right)}{\sqrt{5}}$

Soln: We have, $f(x,zy,z)=\frac{1}{2}x^2-xy+\frac{1}{2}z^2$

$$\therefore \quad \vec{\nabla}f=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{2}x^2-xy+\frac{1}{2}z^2\right)=(x-y)\hat{i}+(-x)\hat{j}+z\hat{k}$$

$$\therefore \quad \vec{\nabla}f\big|_{(1,1,2)}=\left(-\hat{j}+2\hat{k}\right)$$

Therefore, direction of $\vec{\nabla}f$ is $\frac{-\hat{j}+2\hat{k}}{\sqrt{5}}$

Correct option is (b)

Example 23: If $\vec{A}$ and $\vec{B}$ are constant vectors, then $\vec{\nabla}\left[\vec{A}.\left(\vec{B}\times\vec{r}\right)\right]$ is

(a) $\vec{A}.\vec{B}$ (b) $\vec{A}\times\vec{B}$ (c) $\vec{r}$ (d) zero

Soln: $\vec{\nabla}\left[\vec{A}\cdot\left(\vec{B}\times\vec{r}\right)\right]=\vec{\nabla}\left[\vec{r}\cdot\left(\vec{A}\times\vec{B}\right)\right]=\vec{\nabla}\left[\left(\vec{A}\times\vec{B}\right)\cdot\vec{r}\right]$

If $\vec{A}$ and $\vec{B}$ are constant vectors, then $\left(\vec{A}\times\vec{B}\right)$ is also constant vector. For constant vector $\vec{a}$

$$\vec{\nabla}\left(\vec{a}\cdot\vec{r}\right)=\vec{a}$$

Therefore, $\vec{\nabla}\left[\left(\vec{A}\times\vec{B}\right)\cdot\vec{r}\right]=\left(\vec{A}\times\vec{B}\right)$

Correct option is (b)

CAREER ENDEAVOUR

Divergence of a vector field:

Divergence of a continuous differentiable vector point function $\vec{V}$ specified in a vector field is given by,

$$\vec{\nabla}.\vec{V}=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left(V_x\hat{i}+V_y\hat{j}+V_z\hat{k}\right)=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=\text{Scalar quantity}$$

Physical Interpetation: Divergence of $\vec{V}$ at point $P(x, y, z)$ is defined as the outward flux of the vector field $\vec{V}$ per unit volume enclosed by an infinitesimal closed surface surrounding point P.

Note:

(i) If $\vec{\nabla}.\vec{V}=0$, then $\vec{V}$ is known as solenoidal vector field.

(ii) If $\vec{\nabla}.\vec{V}=$ negative, then $\vec{V}$ is known as sink field i.e. vector lines are going inward.

(iii) If $\vec{\nabla}.\vec{V}=$ positive, then $\vec{V}$ is known as source field i.e. vector lines are the going outward.

(iv) $\vec{\nabla}\cdot\left(\vec{U}+\vec{V}\right)=\vec{\nabla}\cdot\vec{U}+\vec{\nabla}\cdot\vec{V}$ (v) $\vec{\nabla}.\left(u\vec{V}\right)=(\vec{\nabla}u)\cdot\vec{V}+u(\vec{\nabla}.\vec{V})$

Example 24: A vector $\vec{A}=(5x+2y)\hat{i}+(3y-z)\hat{j}+(2x-az)\hat{k}$ is solenoidal if the constant a has a value.

(a) 4 (b) –4 (c) 8 (d) –8

Soln: $\vec{A}$ will solenoidal if $\vec{\nabla}\cdot\vec{A}=0\Rightarrow 5+3-a=0\Rightarrow a=8$

Correct option is (c)

Example 25: If $\vec{r}=x\hat{i}+y\hat{j}$, then

(a) $\vec{\nabla}\cdot\vec{r}=0$ and $\vec{\nabla}|\vec{r}|=\vec{r}$ (b) $\vec{\nabla}\cdot\vec{r}=2$ and $\vec{\nabla}|\vec{r}|=\vec{r}$

(c) $\vec{\nabla}\cdot\vec{r}=2$ and $\vec{\nabla}|\vec{r}|=\dfrac{\vec{r}}{r}$ (d) $\vec{\nabla}\cdot\vec{r}=3$ and $\vec{\nabla}|\vec{r}|=\dfrac{\vec{r}}{r}$

Soln: $\vec{r}=x\hat{i}+y\hat{j}\Rightarrow\vec{\nabla}\cdot\vec{r}=2$

$$\vec{\nabla}|\vec{r}|=\vec{\nabla}r=1\cdot\hat{r}=\frac{\vec{r}}{|\vec{r}|}$$

Correct option is (c)

Curl of a vector field:

Curl of a continuous differentiable vector point function $\vec{V}$ specified in a vector field is given by,

$$\vec{\nabla}\times\vec{V}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(V_x\hat{i}+V_y\hat{j}+V_z\hat{k}\right)=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ \dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\\ V_x&V_y&V_z\end{vmatrix}$$

It is the measurement of rotation of vector field and the direction of curl of vector is along axis of rotation.

Note:

(i) If $\vec{\nabla}\times\vec{V}=0$, then $\vec{V}$ is known as an irrotational vector and we can write $\vec{V}=\vec{\nabla}\phi$

(ii) If $\vec{\nabla}\times\vec{V}\neq 0$, then $\vec{V}$ is known as rotational vector.

(iii) $\vec{\nabla}\times(\vec{U}+\vec{V})=\vec{\nabla}\times\vec{U}+\vec{\nabla}\times\vec{V}$

(iv) $\vec{\nabla}\times(u\vec{V})=u\vec{\nabla}\times\vec{V}+(\vec{\nabla}u)\times\vec{V}$

Example 26: The curl of the vector $\vec{A}=z\hat{i}+x\hat{j}+y\hat{k}$ is given by

(a) $\hat{i}+\hat{j}+\hat{k}$ (b) $\hat{i}-\hat{j}+\hat{k}$ (c) $\hat{i}+\hat{j}-\hat{k}$ (d) $-\hat{i}-\hat{j}-\hat{k}$

Soln: $\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ \dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\\ z&x&y\end{vmatrix}=\left(\hat{i}+\hat{j}+\hat{k}\right)$

Correct option is (a)

Example 27: If a vector field $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, then $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right)$ is

(a) 0 (b) $\hat{i}$ (c) $2\hat{j}$ (d) $3\hat{k}$

Soln: $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$

$\vec{\nabla}\times\vec{F} = 0 \Rightarrow \vec{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right) = 0$

Correct option is (a)

Example 28: If a force $\vec{F}$ is derivable from a potential function $V(r)$, where r is the distance from the origin of the coordinate system, it follows that

(a) $\vec{\nabla}\times\vec{F} = 0$ (b) $\vec{\nabla}\cdot\vec{F} = 0$ (c) $\vec{\nabla}V = 0$ (d) $\nabla^2 V = 0$

Soln: If force $\vec{F}$ is derivable from a potential function $V(r)$, then it will be conservative in nature i.e. $\vec{\nabla}\times\vec{F} = 0$

Correct option is (a)

Example 29: A fluid motion is given by

$$\vec{v} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$$

is the motion irrotational? If so, find the velocity potential

Soln: Curl $\vec{v} = \vec{\nabla}\times\vec{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$$

$$= (x\cos z + 2y - x\cos z - 2y)\hat{i} - [y\cos z - y\cos z]\hat{j} + (\sin z - \sin z)\hat{k} = 0$$

Hence, the motion is irrotational.

So, $\vec{v} = \vec{\nabla}\phi$ where ϕ is called velocity potential.

$$\Rightarrow \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) = \vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

Comparing both sides we get,

$$\frac{\partial\phi}{\partial x} = (y\sin z - \sin x) \Rightarrow \phi = xy\sin z + \cos x + f_1(y,z)$$

$$\frac{\partial\phi}{\partial y} = (x\sin z + 2yz) \Rightarrow \phi = xy\sin z + y^2 z + f_1(x,z)$$

$$\frac{\partial\phi}{\partial z} = (xy\cos z + y^2) \Rightarrow \phi = xy\sin z + y^2 z + f_1(x,y)$$

Therefore, the velocity potential will be $\phi = xy\sin z + \cos x + y^2 z + c$

Important Vector Identities:

If ϕ, ψ are scalar point functions and $\vec{A}$, $\vec{B}$ are vector point functions in certain region then.

(1) $\vec{\nabla}(\phi+\psi)=\vec{\nabla}\phi+\vec{\nabla}\psi$

(2) $\vec{\nabla}(\phi\psi)=\phi\vec{\nabla}\psi+\psi\vec{\nabla}\phi$

(3) $\vec{\nabla}\cdot(\vec{A}+\vec{B})=\vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{B}$

(4) $\vec{\nabla}\times(\vec{A}+\vec{B})=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$

(5) $\vec{\nabla}(\vec{A}\cdot\vec{B})=\vec{A}\times(\vec{\nabla}\times\vec{B})+(\vec{A}\cdot\vec{\nabla})\vec{B}+\vec{B}\times(\vec{\nabla}\times\vec{A})+(\vec{B}\cdot\vec{\nabla})\vec{A}$

(6) $\vec{\nabla}\cdot(\phi\vec{A})=\vec{\nabla}\phi\cdot\vec{A}+\phi\left(\vec{\nabla}\cdot\vec{A}\right)$

(7) $\vec{\nabla}\times(\phi\vec{A})=\phi(\vec{\nabla}\times\vec{A})+(\vec{\nabla}\phi)\times\vec{A}$

(8) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$

(9) $\vec{\nabla}\times(\vec{\nabla}\phi)=0$

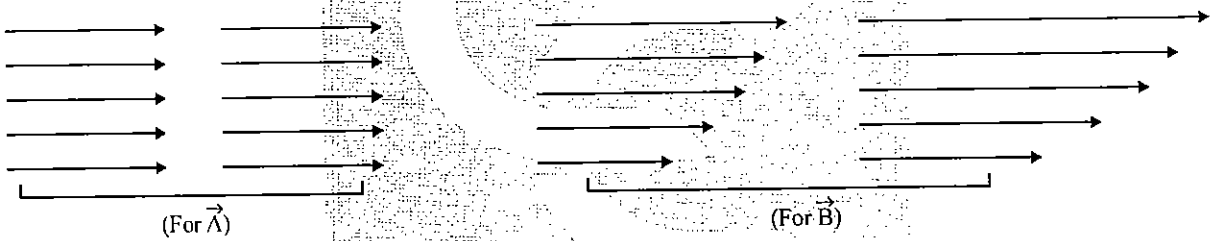
(10) $\vec{\nabla}\times(\vec{A}\times\vec{B})=\vec{B}\cdot(\vec{\nabla}\times\vec{A})-\vec{A}\cdot(\vec{\nabla}\times\vec{B})$

(11) $\vec{\nabla}\times(\vec{A}\times\vec{B})=(\vec{B}\cdot\vec{\nabla})\vec{A}-(\vec{A}\cdot\vec{\nabla})\vec{B}+\vec{A}(\vec{\nabla}\cdot\vec{B})-\vec{B}(\vec{\nabla}\cdot\vec{A})$

(12) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}$

(13) For a constant vector $\vec{A}$, $\vec{\nabla}\left(\vec{A}.\vec{r}\right)=\vec{A}$ and $\vec{\nabla}\times\left(\vec{A}\times\vec{r}\right)=2\vec{A}$

Example 30: Consider the field lines drawn below for two different vector fields $\vec{A}$ and $\vec{B}$



Now choose correct statement(s)

(a) vector field $\vec{A}$ is solenoidal in nature (b) vector field $\vec{A}$ is conservative in nature

(c) vector field $\vec{B}$ is not solenoid in nature (d) vector field $\vec{B}$ is conservative in nature

Soln: The length of the vector $\vec{A}$ is not changing in a direction either parallel or perpendicular to the vector. So,

$$\vec{\nabla}\cdot\vec{A}=0,\ \vec{\nabla}\times\vec{A}=0,$$

Hence, $\vec{A}$ is solenoidal and conservative in nature.

and the length of the vector $\vec{B}$ is changing in a direction both parallel and perpendicular to the vector. So,

$$\vec{\nabla}\cdot\vec{B}\neq 0,\ \vec{\nabla}\times\vec{B}\neq 0,$$

Hence, $\vec{B}$ is neither solenoidal nor conservative in nature.

Correct options are (a), (b) and (c)

Example 31: If $\vec{A}=x\hat{e}_x+y\hat{e}_y+z\hat{e}_z$, then $\nabla^2\vec{A}$

(a) 1 (b) 3 (c) 0 (d) -3

Soln: $\vec{A}=x\hat{e}_x+y\hat{e}_y+z\hat{e}_z$

$$\nabla^2\vec{A}=\frac{\partial^2}{\partial x^2}\left(x\hat{e}_x+y\hat{e}_y+z\hat{e}_z\right)+\frac{\partial^2}{\partial y^2}\left(x\hat{e}_x+y\hat{e}_y+z\hat{e}_z\right)+\frac{\partial^2}{\partial z^2}\left(x\hat{e}_x+y\hat{e}_y+z\hat{e}_z\right)=0$$

Correct option is (c)

Example 32: For a scalar function ϕ satisfying the laplace equation, $\nabla\phi$ has

(a) zero curl and non-zero divergence (b) non-zero curl and zero divergence

(c) zero curl and zero divergence (d) mon-zero curl and non-zero divergence

Soln: $\vec{\nabla}\times\vec{\nabla}\phi=0$. From Laplacian equation $\nabla^2\phi=0 \Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\phi)=0$

Correct option is (c)

Example 33: Show that, $\nabla^2 r^n = n(n+1)r^{n-2}$.

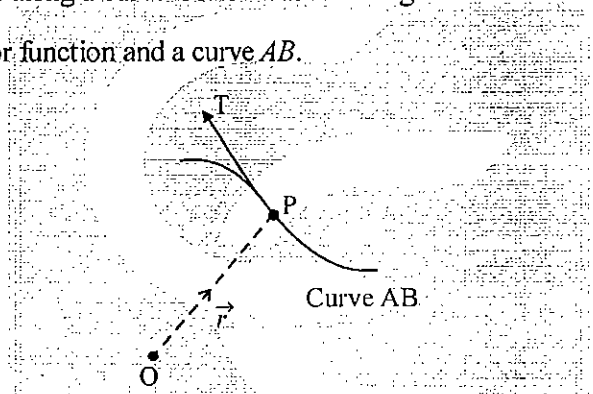
Soln: $\nabla^2 r^n = \vec{\nabla}.\left(\vec{\nabla}r^n\right)=\vec{\nabla}.\left(nr^{n-1}\hat{r}\right)=\vec{\nabla}.\left(nr^{n-2}\vec{r}\right)=\vec{\nabla}\left(nr^{n-2}\right).\vec{r}+nr^{n-2}\left(\vec{\nabla}.\vec{r}\right)$

$=n(n-2)r^{n-3}\hat{r}.\vec{r}+3nr^{n-2}=n(n-2)r^{n-2}+3nr^{n-2}=n(n+1)r^{n-2}$

1.4 Line Integration of Vectors

The integration of a vector along a curve is known as line integration of vectors.

Let $\vec{F}(x,y,z)$ be a vector function and a curve AB.



Line integral of a vector function $\vec{F}$ along the curve AB is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.

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Component of $\vec{F}$ along a tangent PT at P = Dot product of $\vec{F}$ and unit vector along PT

$$= \vec{F}\cdot\frac{d\vec{r}}{ds}\left(\frac{d\vec{r}}{ds}\text{ is a unit vector along tangent PT}\right)$$

Line integral of $\vec{F}$ from A to B along the curve C will be $=\int_c\left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right)ds=\int_c\vec{F}\cdot d\vec{r}$

Note:

(i) If $\vec{F}$ represents the variable force acting on a particle along arc AB, then the total work done

$W_{AB}=\int_A^B\vec{F}\cdot d\vec{r}$

(ii) If $\vec{V}$ represents the velocity of a liquid then $\oint_c\vec{V}\cdot d\vec{r}$ is called the circulation of $\vec{V}$ round closed curve C.

(iii) When the path of integration is a closed curve then notation of integration is $\oint$ in place of $\int$.

(iv) Work done by a conservative field $\vec{A}$ in moving a particle from point P to Q will be

$$\int_P^Q \vec{F}\cdot\overline{dr} = \int_P^Q \vec{\nabla}\phi\cdot\overline{dr} = \int_P^Q d\phi = \phi_Q - \phi_P = \text{independent of path.}$$

(v) Work done by a conservative field $\vec{A}$ in moving a particle around a closed path C is $\oint_c \vec{F}\cdot\overline{dr} = 0$

Example 34: If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, then evaluate the line integral $\oint_C \vec{F}.d\vec{r}$ along the curve C in the x-y plane given by $y = x^3$ from the point (1,1) to (2,8).

Soln: $\vec{r} = x\,\hat{i} + x^3\,\hat{j} \Rightarrow d\vec{r} = dx\,\hat{i} + 3x^2dx\,\hat{j}$

$$So, \int_C \vec{F}.d\vec{r} = \int_1^2 \left[(5x^4 - 6x^2)\hat{i} + (2x^3 - 4x)\hat{j}\right].\left[dx\,\hat{i} + 3x^2dx\,\hat{j}\right]$$

$$= \int_1^2 (6x^5 + 5x^4 - 12x^3 - 6x^2)dx = 35$$

Example 35: If a force field $\vec{A} = y\hat{i} - x\hat{j}$ is applied to a particle, then the work done by the force field in traversing the particle around a circle C in x-y plane, with center at the origin and radius 1 units is equal to (C is traversed in the counter-clockwise direction)

(a) 2π (b) -2π (c) 8π (d) -8π

Soln: $\oint_C \vec{A}.d\vec{r} = \oint_C [ydx - xdy]$

Putting $x = \cos\theta$ and $y = \sin\theta$ we get,

$$= \int_0^{2\pi} \left[\sin\theta(-\sin\theta d\theta) - \cos\theta(\cos\theta d\theta)\right] = -\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\,d\theta = -2\pi$$

Correct option is (b)

Example 36: Evaluate $\int_C \vec{F}.\overline{dr}$, where $F = x^2\hat{i} + y^3\hat{j}$ and curve C is the arc of parabola $y = x^2$ in the x-y plane from (0, 0) to (1, 1).

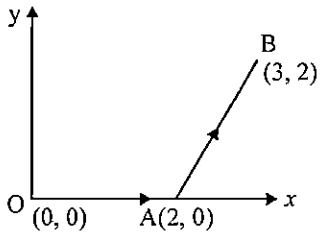
Soln: Along the curve C, $y = x^2 \Rightarrow dy = 2xdx$; So, $\vec{r} = x\hat{i} + x^2\hat{j} \Rightarrow d\vec{r} = dx\hat{i} + 2xdx\hat{j}$

Therefore, $\int_C \vec{F}.d\vec{r} = \int_{x=0}^{1} \left[x^2dx + x^6(2x)dx\right]$

$$= \int_0^1 (x^2 + 2x^7)dx = \left[\frac{x^3}{3} + \frac{2x^8}{8}\right]_0^1 = \frac{7}{12}$$

Example 37: If $\vec{F}=(2x+y)\hat{i}+(3y+x)\hat{j}$, evaluate $\int_C \vec{F}.\overrightarrow{dr}$ where C is the curve in the xy-plane consisting of the straight lines from (0, 0) to (2, 0) and then to (3, 2).

Soln: The path of integration C has been shown in the figure below:



We have, $\int_C \vec{F}.d\vec{r} = \int_C \left[(2x+y)\hat{i}+(3y-x)\hat{j}\right].\left(dx\,\hat{i}+dy\,\hat{j}\right)$

$$=\int_C\left[(2x+y)dx+(3y-x)dy\right]$$

Path 1: Along the straight line OA; $y=0 \Rightarrow dy=0$ and x varies from 0 to 2.

Path 2: Equation of the straight line AB is $y-0=\frac{2-0}{3-2}(x-2)$ *i.e.* $y=2x-4$

Therefore, along AB, $y=2x-4 \Rightarrow dy=2dx$ and x varies from 2 to 3.

$$\therefore \quad \int_C \vec{F}.d\vec{r}=\int_0^2\left[(2x+0)dx+0\right]+\int_2^3\left[(2x+2x-4)dx+(6x-12-x)2dx\right]$$

$$=\left[x^2\right]_0^2+\int_0^2(14x-28)dx=4+14\int_2^3(x-2)dx=11$$

Example 38: Find the total work done in moving a particle in a force field given by $F=3xy\hat{i}-5z\hat{j}+10x\hat{k}$ along the curve $x=t^2+1$, $y=2t^2$, $z=t^3$ from $t=1$ to $t=2$.

Soln: The total work done

$$=\int_C \vec{F}.d\vec{r}=\int_C\left(3xy\hat{i}-5z\hat{j}+10x\hat{k}\right).\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)=\int_C(3xydx-5zdy+10xdz)$$

$$=\int_1^2\left[3\left(t^2+1\right)\left(2t^2\right)(2t)-\left(5t^3\right)(4t)+10\left(t^2+1\right)\left(3t^2\right)\right]dt=\int_1^2\left(12t^5+12t^3-20t^4+30t^4+30t^2\right)dt=303$$

Example 39: Evaluate $\int_C\left\{\left(2xy^3-y^2\cos x\right)dx+\left(1-2y\sin x+3x^2y^2\right)dy\right\}$

Where C is the arc of the parabola $2x=\pi y^2$ from (0, 0) to $\left(\frac{1}{2}\pi,1\right)$

Soln: The given integral in the question is the line integral of the following vector field:

$$\vec{F}=\left(2xy^3-y^2\cos x\right)\hat{i}+\left(1-2y\sin x+3x^2y^2\right)\hat{j}$$

This vector field is conservative in nature as $\vec{\nabla}\times\vec{F}=0$ and it can be written as $\vec{F}=\vec{\nabla}\phi$

$$\frac{\partial\phi}{\partial x}=2xy^3-y^2\cos x \Rightarrow \phi=x^2y^3-y^2\sin x+f_1(y)$$

$$\frac{\partial\phi}{\partial y}=\left(1-2y\sin x+3x^2y^2\right) \Rightarrow \phi=y-y^2\sin x+x^2y^3+f_2(x)$$

Therefore, $\phi = y - y^2 \sin x - x^2 y^3 + c$

Since, work done by a conservative field i.e value of the given integral is independent of path and will be equal to the difference of scalar potential ϕ at initial and final point.

So, $I = \phi\left(\frac{\pi}{2}, 1\right) - \phi(0,0) = \left[y - y^2 \sin x + x^2 y^3\right]_{(0,0)}^{(\pi/2,1)} = \frac{\pi^2}{4}$

Example 40: Suppose an object moves from $(-1, 1)$ to $(2, 4)$ along the path $\vec{r}(t) = \langle t, t^2 \rangle$, subject to the force $\vec{F} = \langle x \sin y, y \rangle$, then the work done by $\vec{F}$ is

(a) $\frac{15}{2} + \frac{\cos(1) - \cos(4)}{2}$ (b) $\frac{15}{2} - \frac{\cos(1) + \cos(4)}{2}$

(c) $\frac{15}{2} + \frac{\sin(1) - \sin(4)}{2}$ (d) $\frac{15}{2} - \frac{\sin(1) + \sin(4)}{2}$

Soln: $\vec{r}(t) = t\hat{i} + t^2\hat{j} \Rightarrow x = t,\ y = t^2$ and $\vec{F} = x \sin y\,\hat{i} + y\,\hat{j} \Rightarrow \vec{F} = t \sin t^2\,\hat{i} + t^2\hat{j}$

Work done $W = \int \vec{F} \cdot d\vec{r}$, Here, $d\vec{r} = \left(\hat{i} + 2t\hat{j}\right)dt$

Therefore, $W = \int_{t=-1}^{2} \left(t \sin t^2 + 2t^3\right)dt = \int_{-1}^{2} t \sin t^2\, dt + \int_{-1}^{2} 2t^3\, dt$

Assuming $t^2 = z$, $2t\,dt = dz$ in the first integration, we get

$$\int_{-1}^{2} t \sin t^2\, dt = \int_{1}^{4} \sin z \frac{dz}{2} = \frac{1}{2}(-\cos z)_1^4 = \frac{1}{2}(\cos 1 - \cos 4)$$

Therefore, $W = \frac{1}{2}(\cos 1 - \cos 4) + 2\left(\frac{t^4}{4}\right)_{-1}^{2} = \frac{1}{2}(\cos 1 - \cos 4) + \frac{1}{2}(15)$

Correct option is (a)

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Example 41: Value of the circulation of the field $\vec{F} = -x^2 y\,\hat{i} + xy^2\,\hat{j} + (y^3 - x^3)\,\hat{k}$ around the curve C, where C is the intersection of the sphere $x^2 + y^2 + z^2 = 25$ and the plane $z = 4$ and is oriented counterclockwise, is

(a) 81π (b) -81π (c) $\frac{81\pi}{4}$ (d) $-\frac{81\pi}{4}$

Soln: Here, $C : x^2 + y^2 = 9$ (curve of intersection of sphere and plane $z = 4$)

$\therefore \quad \oint_C \vec{F} \cdot d\vec{r} = \oint_C -x^2 y\,dx + xy^2\,dy + \left(y^3 - x^3\right)dz$

Let $x = 3\cos\theta \Rightarrow dx = -3\sin\theta\,d\theta$

$y = 3\sin\theta \Rightarrow dy = 3\cos\theta\,d\theta$

and $z = 4 \Rightarrow dz = 0$

$$\oint \vec{F} \cdot d\vec{r} = \int_{\theta=0}^{2\pi} \left(-9\cos^2\theta \times 3\sin\theta \times -3\sin\theta\,d\theta + 3\cos\theta \times 9\sin^2\theta \times 3\cos\theta\,d\theta\right)$$

$$= 81\int_{\theta=0}^{2\pi} \cos^2\theta \sin^2\theta d\theta = \frac{81}{4}\int_{\theta=0}^{2\pi} (\sin 2\theta)^2 d\theta$$

$$= \frac{81}{4}\int_0^{2\pi} \frac{1-\cos 4\theta}{2} d\theta = \frac{81}{8}\left(\theta - \frac{\sin 4\theta}{4}\right)_0^{2\pi} = \frac{81}{8}\times 2\pi = \frac{81\pi}{4}$$

Correct option is (c)

Example 42: If the value of line integral $I = \int_A^B \vec{a}\cdot d\vec{r}$, where $\vec{a} = (xy^2 + z)\hat{i} + (x^2y + 2)\hat{j} + x\hat{k}$, A is the point (c, c, h) and B is the point $\left(2c, \frac{c}{2}, h\right)$ along the path c_1, given by $x = cu$, $y = \frac{c}{u}$, $z = h$ is I_1, and along the path c_2 given by $2y = 3c - x$, $z = h$ is I_2. Then, the ratio I_1/I_2 is equal to (u is a parameter)

(a) $\frac{2c(h-1)}{h}$ (b) $\frac{2h}{c(h-1)}$ (c) 2 (d) 1

Soln: $\vec{a} = \left(xy^2 + z\right)\hat{i} + \left(x^2y + 2\right)\hat{j} + x\hat{k}$

$$\vec{\nabla}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 + z & x^2y + 2 & x \end{vmatrix} = i(0-0) - j(1-1) + \hat{k}(2xy - 2xy) = 0$$

$\because \quad \vec{\nabla}\times\vec{a} = 0$

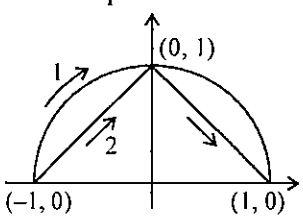
Hence, $\vec{a}$ is a conservative vector field. Therefore, it's line integral will be independent of path and will depend only on initial and final point.

Here, C_1 and C_2 are two different paths taken between the same points A and B. Hence, the line integrals I_1 and I_2 would have same value.

Therefore, $\frac{I_1}{I_2} = 1$

Correct option is (d)

Example 43: Consider the integral $I = \int \frac{x\,dy - y\,dx}{x^2 + y^2}$. Then which of the following statement(s) is/are correct about the value of I along each of the two paths indicated in figure :



(a) For the path-1, $I_1 = -\pi$ (b) For the path-1, $I_1 = \pi$

(c) For the path-2, $I_2 = -\pi$ (d) For the path-2, $I_2 = \pi$

Soln: $I = \int \frac{xdy - ydx}{x^2 + y^2} = \int \left(\frac{x\hat{j} - y\hat{i}}{x^2 + y^2} \right) \cdot d\vec{r} = \int \vec{a} . d\vec{r}$

$\because \quad \vec{\nabla} \times \vec{a} = 0$

Hence, $\vec{a}$ is a conservative vector field. Therefore, it's line integral will be independent of path and will depend only on initial and final point.

Here, Path 1 & Path 2 are two different paths taken between the same points (-1, 0) and (1, 0). Hence, the line integrals I_1 and I_2 would have same value.

Along path 1: (Semi circle of radius 1 unit)

Put $x = \cos\theta \Rightarrow dx = -\sin\theta d\theta$ and $y = \sin\theta \Rightarrow dy = \cos\theta d\theta$

$$\therefore \quad I_1 = \int_{\theta=\pi}^{0} \frac{(\cos\theta \times \cos\theta - \sin\theta \times -\sin\theta) d\theta}{\sin^2\theta + \cos^2\theta}$$

$$I_1 = \int_{\pi}^{0} d\theta = -\pi = I_2$$

Correct options are (a) and (c)

Example 44: Let a vector $\vec{V}$ is given by $\vec{V} = \frac{\vec{r}}{r^3} (r \neq 0)$

Which one of the following is/are CORRECT?

(a) Value of the line integral of this vector around any closed curve is zero

(b) This vector can be written as the gradient of some scalar function

(c) This is a conservative vector field.

(d) This vector can represent the magnetic field of some current distribution.

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Soln: Here, $\vec{\nabla} \times \left(\frac{\vec{r}}{r^3} \right) = 0$

Therefore, $\vec{V} = \frac{\vec{r}}{r^3}$ is a conservative field.

So, the value of the line integral of this vector around any closed curve C is zero i.e. $\oint_C \vec{V} \cdot d\vec{r} = 0$,

Vector can be written as the gradient of some scalar function $\vec{V} = \vec{\nabla}\phi$.

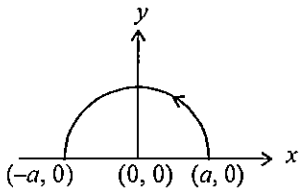
Since, magnetic field $\vec{B}$ is not conservative, therefore this vector cannot represent the magnetic field of some current distribution.

Correct option is (a,b,c)

Example 45: The value of the integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \frac{y^2}{x^2 + y^2}\hat{i} - \frac{x^2}{x^2 + y^2}\hat{j}$ and $C : y = \sqrt{a^2 - x^2}$ is _____ $\times a^3$.

Soln: Given curve C is a half circle above x-axis

Hence, $\oint_C \vec{F} \cdot d\vec{r} = \int_C \frac{y^2 dx}{x^2 + y^2} - \frac{x^2}{x^2 + y^2} dy$



Put $x = a\cos\theta \Rightarrow dx = -a\sin\theta d\theta$ and $y = a\sin\theta \Rightarrow dy = a\cos\theta d\theta$

$$\therefore \quad \int \vec{F}\cdot d\vec{r} = \int_{\theta=0}^{\pi}\left(a^2\sin^2\theta \times -a\sin\theta\right)d\theta - \left(a^2\cos^2\theta \times a\cos\theta\right)d\theta = -a^3\int_{\theta=0}^{\pi}\left(\sin^3\theta + \cos^3\theta\right)d\theta$$

$$= -a^3\left[\int_{\theta=0}^{\pi}\frac{1}{4}\left(3\sin\theta - \sin 3\theta\right)d\theta + \int_{0}^{\pi}\frac{1}{4}\left(\cos 3\theta - 3\cos\theta\right)d\theta\right]$$

$$= -\frac{a^3}{4}\left[3\times -\cos\theta + \frac{\cos 3\theta}{3} + \frac{\sin 3\theta}{3} - \frac{3\sin\theta}{3}\right]_0^{\pi}$$

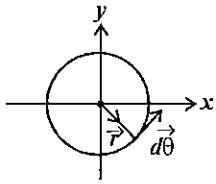
$$= -\frac{a^3}{4}\left[3\frac{1}{3} + 3 - \frac{1}{3}\right] = -\frac{a^3}{4}\left[\frac{16}{3}\right] = -\frac{4a^3}{3}$$

Correct answer is (1.33)

Example 46: The value of contour integral $\left|\int_C \vec{r}\times d\vec{\theta}\right|$, for a circle C of radius r with centre at the origin is

(a) $2\pi r$ (b) $\frac{r^2}{2}$ (c) πr^2 (d) r

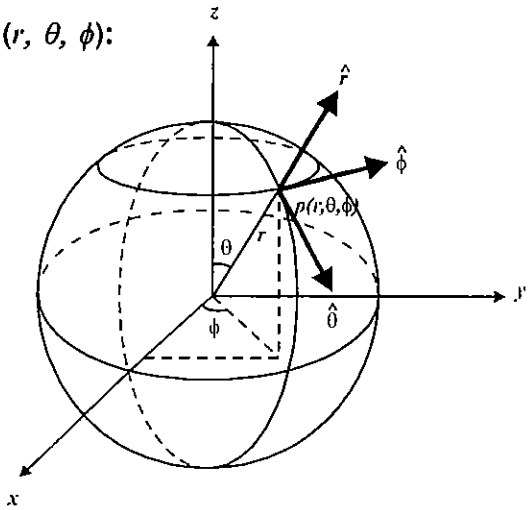
Soln: $\left|\int_C \vec{r}\times d\vec{\theta}\right| = \int_C r d\theta \sin 90^\circ = r\int_0^{2\pi} d\theta = 2\pi r$



Correct option is (a)

1.5 Orthogonal Currvilinear Co-ordinates

(A) Spherical Polar co-ordinates (r, θ, ϕ):



Let the point P (in the above figure) has cartesian coordinates (x, y, z) and spherical polar coordinates (r, θ, ϕ).

Relation between cartesian and spherical polar co-ordiantes:

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta$$

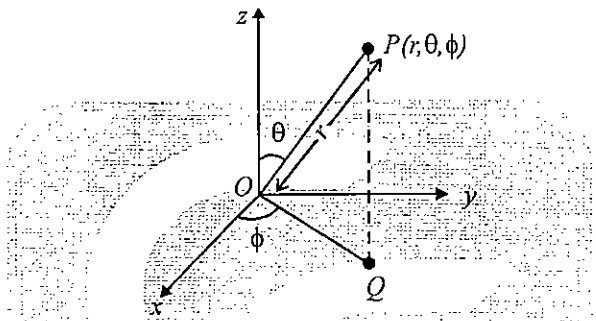
In spherical polar co-ordinates,

$r = \sqrt{x^2 + y^2 + z^2}$ = radial distance of the point P from the origin

$\theta = \tan^{-1}\dfrac{\sqrt{x^2 + y^2}}{z}$ = angle which OP makes with z-axis

$\phi = \tan^{-1}\left(\dfrac{y}{x}\right)$ = angle which OQ (projection of OP in x-y plane) makes with positive x-axis

Corresponding limits: $0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$



Unit vectors in Spherical Polar coordinates:

The position vector of P is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Putting (x, y, z) in the above relations,

$$\vec{r} = (r\sin\theta\cos\phi)\hat{i} + (r\sin\theta\sin\phi)\hat{j} + r\cos\theta\hat{k}$$

The unit vector along r is given by:

$$\hat{r} = \frac{\vec{r}}{|r|} = \frac{r\sin\theta\cos\phi\hat{i} + r\sin\theta\sin\phi\hat{j} + r\cos\theta\hat{k}}{r}$$

$$\Rightarrow \quad \boxed{\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}}$$

The unit vector along θ is given by

$$\hat{\theta} = \frac{\partial\vec{r}/\partial\theta}{|\partial r/\partial\theta|} = \frac{r\cos\theta\cos\phi\hat{i} + r\cos\theta\sin\phi\hat{j} + r\sin\theta\hat{k}}{\sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}}$$

$$\Rightarrow \quad \boxed{\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}}$$

The unit vector along ϕ is given by

$$\hat{\phi} = \frac{\partial\vec{r}/\partial\phi}{|\partial r/\partial\phi|} = \frac{-r\sin\theta\sin\phi\hat{i} + r\sin\theta\cos\phi\hat{j}}{\sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}}$$

$\Rightarrow$ $\boxed{\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}}$

Note: The unit vector $\hat{r}, \hat{\theta}, \hat{\phi}$ are orthogonal vectors and it thus follows,

$$\boxed{\hat{r}\times\hat{\theta} = \hat{\phi},\ \hat{\theta}\times\hat{\phi} = \hat{r}, \hat{\phi}\times\hat{r} = \hat{\theta}}$$

Line elements:

Line element between r to $r+dr$: $d\vec{l}_r = dr\,\hat{r}$ (along r curve)

Line element between θ to $\theta + d\theta$: $d\vec{l}_\theta = rd\theta\,\hat{\theta}$ (along θ curve)

Line element between ϕ to $\phi + d\phi$: $d\vec{l}_\phi = r\sin\theta d\phi\,\hat{\phi}$ (along ϕ curve)

Surface elements:

Constant r surface : $d\vec{s}_r = d\vec{l}_\theta \times d\vec{l}_\phi = r^2\sin\theta d\theta d\phi\,\hat{r}$ (surface of a sphere)

Constant θ surafce : $d\vec{s}_\theta = d\vec{l}_\phi \times d\vec{l}_r = r\sin\theta dr d\phi\,\hat{\theta}$

Constant ϕ surface : $d\vec{s}_\phi = d\vec{l}_r \times d\vec{l}_\theta = rdrd\theta\,\hat{\phi}$

Volume element:

$$dV = d\vec{l}_r.\left(d\vec{l}_\theta \times d\vec{l}_\phi\right) = r^2\sin\theta dr d\theta d\phi$$

Gradient: $\vec{\nabla} = \dfrac{\partial}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial}{\partial\theta}\hat{\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\phi}\hat{\phi}$

Divergence: $\vec{\nabla}.\vec{a} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 a_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta a_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial a_\phi}{\partial\phi}$

Curl: $\vec{\nabla}\times\vec{a} = \dfrac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ a_r & ra_\theta & r\sin\theta a_\phi \end{vmatrix}$

Laplacian: $\nabla^2 = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial}{\partial r}\right) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\theta}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial^2}{\partial\phi^2}$

Conversion of a vector from Cartesian to Spherical polar co-ordiantes:

$$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\hat{k}$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{bmatrix}$$

Suppose the vector is given in cartesian coordinates i.e. $\vec{r}=a_x\hat{i}+a_y\hat{j}+a_z\hat{k}$ and we have to convert it in sphereical polar coordinates i.e $\vec{r}=a_r\hat{r}+a_\theta\hat{\theta}+a_\phi\hat{\phi}$, the components are related by the following equation.

i.e. $$\begin{bmatrix}a_r\\ a_\theta\\ a_\phi\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}a_x\\ a_y\\ a_z\end{bmatrix}$$

Limits for various types of integrations:

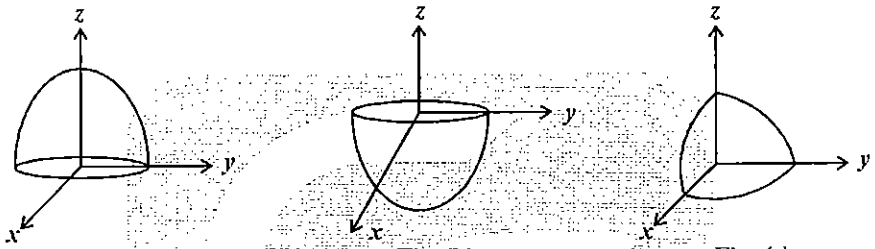


Fig. (a) Upper hemisphere

Fig. (b) Lower hemisphere

Fig. (c) First Octant of sphere

Case-I: Full sphere $x^2+y^2+z^2=a^2$

Limit of volume integration : $0\le r\le a;\ 0\le\theta\le\pi;\ 0\le\phi\le 2\pi$

Limit of surface integration : $r=\text{const}=a;\ 0\le\theta\le\pi;\ 0\le\phi\le 2\pi$

Case-II: Upper Hemisphere $x^2+y^2+z^2=a^2, z>0$

Limit of volume integration : $0\le r\le a;\ 0<\theta\le\frac{\pi}{2};\ 0\le\phi\le 2\pi$

Limit of surface integration : $r=\text{const}=a;\ 0\le\theta\le\frac{\pi}{2};\ 0\le\phi\le 2\pi$

Case-III: Lower hemisphere $x^2+y^2+z^2=a^2, z<0$

Limit of volume integration : $0\le r\le a;\ \frac{\pi}{2}<\theta\le\pi;\ 0\le\phi\le 2\pi$

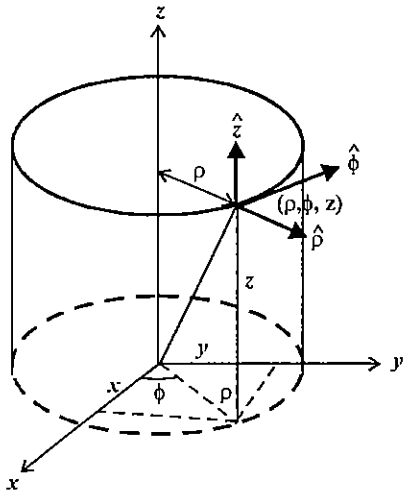
Limit of surface integration : $r=\text{const}=a;\ \frac{\pi}{2}\le\theta\le\pi;\ 0\le\phi\le 2\pi$

Case-IV: First octant of sphere $x^2+y^2+z^2=a^2,\ x,y,z>0$

Limit of volume integration : $0\le r\le a;\ 0<\theta\le\frac{\pi}{2};\ 0\le\phi\le\frac{\pi}{2}$

Limit of surface integration : $r=\text{const}=a;\ 0\le\theta\le\frac{\pi}{2};\ 0\le\phi\le\frac{\pi}{2}$

(B) Cylindrical co-ordinates (ρ, ϕ, z):



Let the point P (in above figure) has cartesian coordinates (x, y, z) and cylinderical coordinates (ρ, ϕ, z).

So, the relation between cartesian and cyllindrical co-ordiantes:

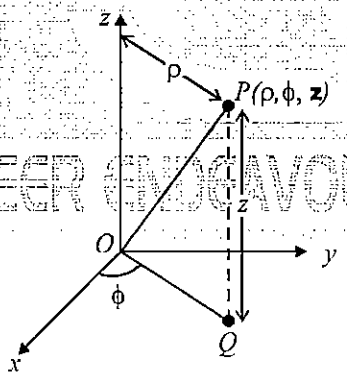
$$x = \rho \cos\phi,\ y = \rho \sin\phi,\ z = z$$

In cylindrical co-ordinates,

$\rho = \sqrt{x^2 + y^2}$ = perpendicular distance point 'P' from the z-axis

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$ = angle which OQ (projection of OP in x-y plane) makes with positive x-axis

z = height of the point 'P' above the x-y plane



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Unit Vector in cylindrical coordinates:

The position vector of the point P is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Putting the value of (x, y, z) in the above relation

$$\vec{r} = \rho\cos\phi\hat{i} + \rho\sin\phi\hat{j} + z\hat{k}$$

The unit vector along ρ is given by

$$\hat{\rho} = \frac{\partial\vec{r}/\partial\rho}{|\partial\vec{r}/\partial\rho|} = \cos\phi\hat{i} + \sin\phi\hat{j} \quad \Rightarrow \quad \boxed{\hat{\rho} = \cos\phi\hat{i} + \sin\phi\hat{j}}$$

The unit vector along ϕ is given by

$$\hat{\phi} = \frac{\partial \vec{r}/\partial \phi}{|\partial \vec{r}/\partial \phi|} = -\sin\phi\hat{i} + \cos\phi\hat{j} \quad \Rightarrow \quad \boxed{\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}}$$

The unit vector along z is given by

$$\hat{z} = \frac{\partial \vec{r}/\partial z}{|\partial \vec{r}/\partial z|} = \hat{k} \quad \Rightarrow \quad \boxed{\hat{z} = \hat{k}}$$

Note: $\hat{\rho}, \hat{\phi}$ and $\hat{z}$ are orthogonal to each other and they follow

$$\boxed{\hat{\rho} \times \hat{\phi} = \hat{z},\ \hat{\phi} \times \hat{z} = \hat{\rho},\quad \hat{z} \times \hat{\rho} = \hat{\phi}}$$

Line elements:

Line element between ρ to $\rho + d\rho$: $d\vec{l}_\rho = d\rho\ \hat{\rho}$ (along ρ curve)

Line element between ϕ to $\phi + d\phi$: $d\vec{l}_\phi = \rho d\phi\ \hat{\phi}$ (along ϕ curve)

Line element between z to $z + dz$: $d\vec{l}_z = dz\ \hat{z}$ (along z curve)

Surface elements:

Constant ρ surface : $d\vec{s}_\rho = d\vec{l}_\phi \times d\vec{l}_z = \rho d\phi dz\ \hat{\rho}$ (curved surface of a circular cyllinder)

Constant ϕ surface : $d\vec{s}_\phi = d\vec{l}_z \times d\vec{l}_r = d\rho dz\ \hat{\phi}$

Constant z surafce : $d\vec{s}_z = d\vec{l}_\rho \times d\vec{l}_\phi = \rho d\rho d\phi\ \hat{z}$ (plane surface of a circular cyllinder parallel to x - y plane)

Volume element :

$$dV = d\vec{l}_\rho . \left(d\vec{l}_\phi \times d\vec{l}_z\right) = \rho d\rho d\phi dz$$

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Gradient: $\vec{\nabla} = \frac{\partial}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}$

Divergence: $\vec{\nabla}.\vec{a} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial \phi} + \frac{\partial a_z}{\partial z}$

Curl: $\vec{\nabla} \times \vec{a} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ a_\rho & \rho a_\phi & a_z \end{vmatrix}$

Laplacian: $\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$

Conversion of a vector from Cartesian to Cylindrical co-ordiantes:

$$\hat{\rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\hat{z} = \hat{k}$$

Suppose the vector is given in cartesian coordinates i.e. $\vec{r} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and we have to convert it in sphereical polar coordinates i.e $\vec{r} = a_\rho\hat{\rho} + a_\phi\hat{\phi} + a_z\hat{z}$, the components are related by the following equation.

$$\begin{bmatrix} a_\rho \\ a_\phi \\ a_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

Example 47: Four forces are given below in Cartesian and spherical polar coordinates.

(i) $\vec{F}_1 = K\exp\left(\frac{-r^2}{R^2}\right)\hat{r}$ (ii) $\vec{F}_2 = K\left(x^3\hat{y} - y^3\hat{z}\right)$ (iii) $\vec{F}_3 = K\left(x^3\hat{x} + y^3\hat{y}\right)$ (iv) $\vec{F}_4 = K\left(\frac{\hat{\phi}}{r}\right)$

where K is a constant. Identify the correct option.
(a) (iii) and (iv) are conservative but (i) and (ii) are not
(b) (i) and (ii) are conservative but (iii) and (iv) are not
(c) (ii) and (iii) are conservative but (i) and (iv) are not
(d) (i) and (iii) are conservative but (ii) and (iv) are not

Soln: $$\vec{\nabla}\times\vec{F}_1 = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ k.\exp\left(-r^2/R^2\right) & 0 & 0 \end{vmatrix} = 0$$

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$$\vec{\nabla}\times\vec{F}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & kx^3 & -kz^3 \end{vmatrix} = 3kx^2\hat{k} \Rightarrow \vec{\nabla}\times\vec{F}_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kx^3 & ky^3 & 0 \end{vmatrix} = 0$$

$$\vec{\nabla}\times\vec{F}_4 = \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & \frac{k}{r} \end{vmatrix} = \frac{1}{r^2\sin\theta}r\hat{\theta}\left[-\frac{k}{r^2}\right] = -\frac{k}{r^3\sin\theta}\hat{\theta}$$

Correct option is (d)

Example 48: Consider two vector functions $\vec{A} = r^n \vec{r}$ and $\vec{B} = \frac{\vec{r}}{r}$ (where r is a spherical co-ordinate)

Which of the following statements is/are CORRECT?

(a) Vector function $\vec{A}$ can be written as $\vec{A} = grad(\phi)$ (where ϕ is any scalar function)

(b) $\nabla^2(A) = (n+1)(n+2)r^{n-1}$, where ∇^2 is a laplacian operator

(c) curl of $\vec{B}$ is equal to zero

(d) vector function $\vec{B}$ is conservative in nature

Soln: $\vec{A} = r^n \vec{r} = r^{n+1}\hat{r}$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ r^{n+1} & 0 & 0 \end{vmatrix}$$

$\boxed{\vec{\nabla} \times \vec{A} = 0}$ i.e. $\vec{A}$ is conservative and $\vec{A}$ can be written as grad (ϕ) i.e. $\vec{A} = grad(\phi)$

$$\nabla^2(A) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(r^{n+1}\right)\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \times (n+1)r^n\right)$$

$$= \frac{(n+1)}{r^2}\frac{\partial}{\partial r}\left(r^{n+2}\right) = \frac{(n+1)}{r^2}(n+2)r^{n+1} = (n+1)(n+2)r^{n-1}$$

$curl\ \vec{B} = \vec{\nabla} \times \vec{B} = 0$

Hence, $\vec{B}$ is conservative in nature.

Correct options are (a), (b), (c) and (d)

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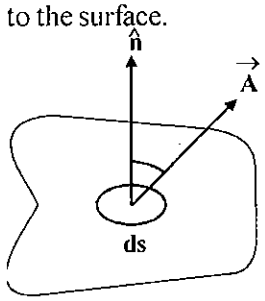
1.6 Surface Integration of Vectors

The integration of vector on an open or closed surface is known as surface integration of vectors.

Let $\vec{A}$ be a vector function and S be the given surface. Surface integral of a vector function $\vec{A}$ over the surface S is defined as the integral of the components of $\vec{A}$ along the normal to the surface.



Components of $\vec{A}$ along the normal

$= \vec{A}.\hat{n}$, where $\hat{n}$ is the unit normal vector to an element ds

Surface integral of $\vec{A}$ over S

$$= \Sigma \vec{A}.\hat{n} = \iint_s \left(\vec{A}.\hat{n}\right) ds$$

Physical Interpetation:

Example 49: Evaluate $\iint_s \vec{F}.\hat{n}\overrightarrow{ds}$, where $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ and S is that part of the surface of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant.

Soln: For the surface of a sphere $(r = \text{radius} = 1 \text{ unit})$,

$$d\vec{S} = \hat{r}\left(r^2 \sin\theta\, d\theta d\varphi\right) = \hat{r}\left(\sin\theta\, d\theta d\varphi\right) = \left(\sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\hat{j} + \cos\theta\hat{k}\right)\left(\sin\theta\, d\theta d\varphi\right)$$

Hence, $\iint_S \vec{F}.\hat{n}dS = \iint_S \vec{F}.d\vec{S} = 3\int_{\theta=0}^{\pi/2}\int_{\varphi=0}^{\pi/2} \sin^3\theta\cos\theta\sin\varphi\cos\varphi\, d\theta\, d\varphi = 3\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{3}{8}$

Example 50: Evaluate $\iint_s \vec{F}.\hat{n}\overrightarrow{ds}$, where $\vec{F} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5

Soln: For the curved surface of the cylinder $(\rho = \text{base radius} = 4 \text{ units})$,

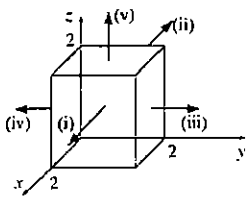
$$d\vec{S} = \hat{\rho}\left(\rho\, d\varphi\, dz\right) = \hat{\rho}\left(4\, d\varphi.dz\right) = \left(\cos\varphi\,\hat{i} + \sin\varphi\hat{j}\right)\left(4\, d\varphi\, dz\right)$$

Hence, $\iint_S \vec{F}.d\vec{S} = 4\int_{z=0}^{5}\int_{\varphi=0}^{2\pi} \left(z\cos\varphi + 4\cos\varphi\sin\varphi\right)dz\, d\varphi = 4\left(\frac{25}{2}\right)(1) + 16(5)(1) = 90$

Example 51: Calculate the surface integral of $\vec{v} = 2xz\,\hat{x} + (x+2)\hat{y} + y\left(z^2 - 3\right)\hat{z}$ over five sides (excluding the bottom) of the cubical box shown in figure.

Soln: (i) For the surface $x = 2$, $d\vec{s} = dy\, dz\, \hat{x}$, $\vec{v}.d\vec{s} = 2xzdy\, dz = 4z\, dy\, dz$, So, $\int \vec{v}.d\vec{s} = 0$

(ii) For the surface $x = 0$, $d\vec{s} = -dy\, dz\, \hat{x}$, $\vec{v}.d\vec{s} = -2xzdy\, dz = 0$, so $\int \vec{v}.d\vec{s} = 0$



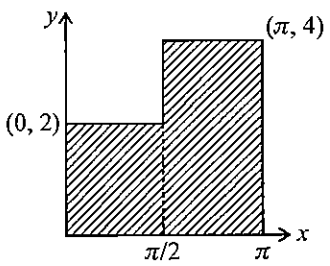
(iii) For the surface $y = 2$, $d\vec{s} = dx\, dz\, \hat{y}$, $\vec{v}.d\vec{s} = (x+2)\, dx\, dz$, so $\int \vec{v}.d\vec{s} = \int_0^2 (x+2)\, dx \int_0^2 dz = 12$

(iv) For the surface $y = 0$, $d\vec{s} = -dx\, dz\, \hat{y}$, $\vec{v}.d\vec{s} = -(x+2)\, dx\, dz$, so $\int \vec{v}.d\vec{s} = -\int_0^2 (x+2)\, dx \int_0^2 dz = -12$

(v) For the surface z = 2, $d\vec{s} = dxdy\, \hat{z}$, $\vec{v}.d\vec{s} = ydxdy$, so $\int \vec{v}.d\vec{s} = \int_0^2 dx \int_0^2 ydy = 4$

Total flux $= \int_{surface} \vec{v}.d\vec{s} = 16 + 0 + 12 - 12 + 4 = 20$

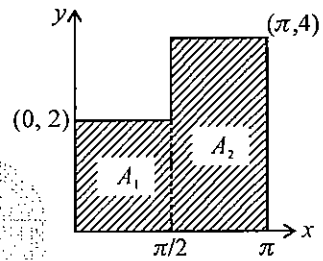
Example 52: The value of integral $\iint_A y\,dx\,dy$, where A is the area as shown in figure :



(a) π (b) 2π (c) 4π (d) 5π

Soln: Let us divide the given area into two parts

$$\iint_A ydxdy = \iint_{A_1} ydxdy + \iint_{A_2} ydxdy$$



$$= \int_{y=0}^{2}\int_{x=0}^{\pi/2} ydxdy + \int_{y=0}^{4}\int_{x=\pi/2}^{\pi} ydxdy$$

$$= \int_{y=0}^{2} (xy)_0^{\pi/2}\,dy + \int_{y=0}^{4} (xy)_{\pi/2}^{\pi}\,dy$$

$$= \frac{\pi}{2}\left(\frac{y^2}{2}\right)_0^2 + \frac{\pi}{2}\left(\frac{y^2}{2}\right)_0^4 = \frac{\pi}{2}\times 2 + \frac{\pi}{2}\times 8 = 10\frac{\pi}{2} = 5\pi$$

Correct option is (d)

Example 53: Consider the vector function $\vec{V} = \rho(2+\sin^2\phi)\hat{\rho} + \rho\sin\phi\cos\phi\hat{\phi} + 3z\hat{z}$ (ρ, ϕ, z are cylindrical co-ordinates), which of the following statement(s) is/are CORRECT ?

(a) $\vec{\nabla}\times\vec{V}\big|_{\rho=1,\,\phi=\frac{\pi}{2},\,z=2}$ is equal to zero

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(b) Flux of $\vec{V}$ through the curved surface of cylinder $x^2 + y^2 = 4,\ 0 \le z \le 5$ in first quadrant is 25π

(c) $\vec{\nabla}\times\vec{V}\big|_{\rho=1,\,\phi=\frac{\pi}{2},\,z=2}$ is $3\hat{\rho} + 2\hat{\phi} + \hat{z}$

(d) Flux of $\vec{V}$ through the curved surface of cylinder $x^2 + y^2 = 4,\ 0 \le z \le 5$ in first quadrant is zero

Soln: $\vec{V} = \rho\left(2+\sin^2\phi\right)\hat{\rho} + \rho\sin\phi\cos\phi\hat{\phi} + 3z\hat{z}$

$$\vec{\nabla}\times\vec{V} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ \rho\left(2+\sin^2\phi\right) & \rho\cdot\rho\sin\phi\cos\phi & 3z \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\hat{\rho}(0-0) - \rho\hat{\phi}(0-0) + \hat{z}\left(2\rho\sin\phi\cos\phi - \rho\times 2\sin\phi\cos\phi\right)\right] = 0$$

Flux of $\vec{V}$ through the curved surface of cylinder $x^2 + y^2 = 4,\ 0 \le z \le 5$ (i.e. $d\vec{S}_\rho = \rho d\phi dz\hat{\rho}$) in first quadrant, is

$$\therefore \quad \iint_S \vec{V}\cdot d\vec{S} = \int_{z=0}^{5}\int_{\phi=0}^{\pi/2} \rho\left(2+\sin^2\phi\right)\times \rho d\phi dz \qquad (\text{Here, } \rho=2)$$

$$= 4\int_{z=0}^{5}\int_{\phi=0}^{\pi/2}\left(2d\phi + \sin^2\phi d\phi\right)dz = 4\int_{z=0}^{5}\left(2\times\frac{\pi}{2}+\frac{\pi}{4}\right)dz$$

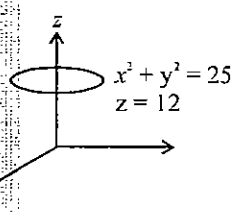
$$= 4\times 5\frac{\pi}{4}\times 5 = 25\pi$$

Correct options are (a), (b)

Example 54: If $\vec{r}$ is the position vector field, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then the value of surface integral $\iint_S \vec{r}\cdot d\vec{S}$, where S is the surface $z = 12,\ x^2 + y^2 \le 25$ is _____ $\times\pi$.

Soln: Here, $d\vec{S} = dxdy\hat{k}$

$$\therefore \quad \iint_S \vec{r}\cdot d\vec{S} = \iint zdxdy = 12\iint dxdy = 12\times\pi(5)^2 = 300\,\pi$$



Correct answer is (300)

Example 55: The area of the part of the cone $z^2 = 3(x^2 + y^2)$ (above xy-plane), which is inside the sphere $x^2 + y^2 + z^2 = 16$ is _____ $\times\pi$.

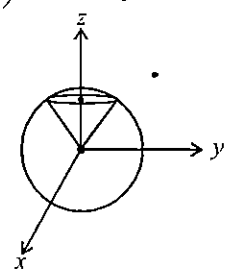
Soln: In spherical polar co-ordinates, cone is the surface for which θ is constant.

Therefore, area of cone,

$$= \iint_{S_\theta} dS_\theta = \iint r\sin\theta dr d\phi$$

For the cone, $z^2 = 3\left(x^2 + y^2\right)$, $\quad \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = 30^\circ = \dfrac{\pi}{6}$

Therefore, area $= \left(\sin\dfrac{\pi}{6}\right)\int_{r=0}^{4} rdr \int_{\phi=0}^{2\pi} d\phi$



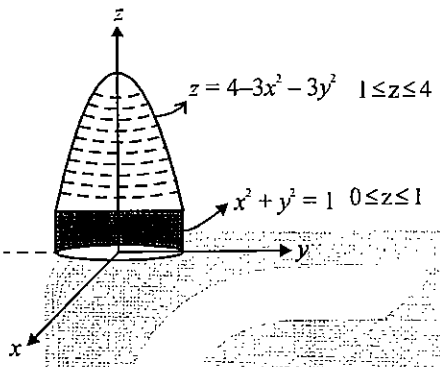
$$= \frac{1}{2}\times\left(\frac{r^2}{2}\right)_0^4(2\pi) = \frac{1}{2}\times\frac{16}{2}\times 2\pi = 8\pi$$

Correct answer is (8)

Example 56: Value of the integral $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = xy\,\hat{i} - \frac{1}{2}y^2\,\hat{j} + z\,\hat{k}$ and the surface consists of the three surfaces $z = 4 - 3x^2 - 3y^2$, $1 \le z \le 4$ on the top, $x^2 + y^2 = 1$, $0 \le z \le 1$ on the sides and $z = 0$ on the bottom, is _____ $\times \pi$.

Soln: $\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \left(\vec{\nabla}\cdot\vec{F}\right) dV$ (since, surface enclosed the volume, shown in the figure below)

$$\vec{\nabla}\cdot\vec{F} = \left(y - \frac{1}{2}\times 2y + 1\right) = 1$$



$$\therefore \quad \iint_S \vec{F}\cdot d\vec{S} = \iint \int_{z=0}^{4-3x^2-3y^2} 1\,dV = \iint \int_{z=0}^{4-3x^2-3y^2} dz\,dx\,dy = \iint \left(4 - 3x^2 - 3y^2\right) dx\,dy$$

In x-y plane, $x^2 + y^2 = 1$, Put $x = r\cos\theta$ and $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \left(4 - 3r^2\right) r\,dr\,d\theta = 2\pi\left(\frac{4r^2}{2} - \frac{3r^4}{4}\right)_0^1 = \frac{5}{2}\pi$$

Correct answer is (2.5)

Example 57: The flux of vector field $\vec{V} = \left(r\sin\theta\,\hat{r} + \hat{\theta} + \hat{\phi}\right)$ through hemispherical bowl of radius one, base on x-y plane and the origin at the centre of the base is _____ $\times \pi^2$

Soln: Flux $= \iint_{S_r} \vec{V}\cdot d\vec{S}_r$

Here, S_r is the curved surface of hemipsherical bowl and $d\vec{S}_r = r^2\sin\theta\,d\theta\,d\phi\,\hat{r}$

$$\therefore \quad \text{Flux} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \left(r\sin\theta\right) r^2\sin\theta\,d\theta\,d\phi$$

$$= r^3 \int_0^{\pi/2} \sin^2\theta\,d\theta \int_0^{2\pi} d\phi = (1)^3 \int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\,d\theta\,(2\pi)$$

$$= 2\pi\left(\frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right)_0^{\pi/2} = 2\pi\times\frac{\pi}{4} = \frac{\pi^2}{2}$$

Correct answer is (0.5)

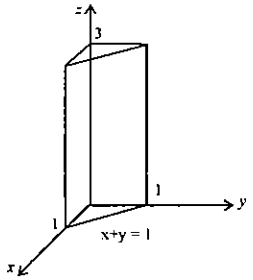
1.7 Volume Integration of Vectors

Volume integral of a vector field $\vec{A}$ within the volume V can be written as,

$$\iiint_V \vec{A}\,dV, \text{ where, } dV \text{ is the infinitesimal volume element}$$

Physical Interpetation:

Example 58: Calculate the volume integral of $T = xyz^2$ over the prism in figure.



Soln: $\int T dv = \int_0^3 z^2 \left\{ \int_0^1 y \left[\int_0^{1-y} x dx \right] dy \right\} dz = \frac{1}{2}\int_0^3 z^2 dz \int_0^1 (1-y)^2 y\, dy = \frac{1}{2}(9)\left(\frac{1}{12}\right) = \frac{3}{8}$

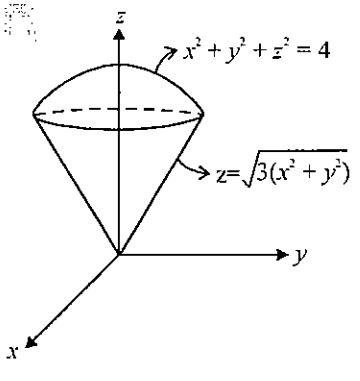
Example 59: The volume of the portion of the cyllinder $x^2 + y^2 = 4$ in the first octant between the planes z = 0 and $3x - z = 0$ is

(a) 2 (b) 4 (c) 8 (d) 16

Soln: Volume of the portion of the cyllinder in the first octant $= V = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{z=0}^{3x} dx dy dz = 8$

Example 60: Let 'u' be the 'ice cream cone' bounded below by $z = \sqrt{3(x^2+y^2)}$ and above by $x^2 + y^2 + z^2 = 4$. Then the volume of 'u' is $\times\pi$

Soln. Volume of 'ice-cream' cone $= \iiint_u dV$



Using spherical co-ordinates

$$\iiint_u dV = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/6} \int_{r=0}^{2} r^2 \sin\theta\, dr d\theta d\phi$$

$$= \left(\frac{r^3}{3}\right)^2 \left(-\cos\theta\right)_0^{\pi/6} (2\pi) = \frac{8}{3} \times \left(-\frac{\sqrt{3}}{2} + 1\right) 2\pi = 0.715\,\pi$$

Correct answer is (0.715)

Example 61: The volume integral of the function

$A(r,\theta,\phi)=r^2\cos\theta\cos\phi$ over the region $0\le r\le 1,\ 0\le\theta\le\frac{\pi}{2},\ 0\le\phi\le\pi$ is equal to _____.

Soln: Volume integral=

$$\iiint AdV=\int_{\phi=0}^{\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^2\cos\theta\cos\phi r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=\int_0^1 r^4dr\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^{\pi}\cos\phi\,d\phi$$

$$=\left(\frac{r^5}{5}\right)_0^1\left(\frac{1}{2}\times\frac{-\cos 2\theta}{2}\right)_0^{\pi/2}(\sin\theta)_0^{\pi}=0$$

Correct answer is (0)

Example 62: Let D denote the solid bounded below by the plane $z+y=2$, above by the cylinder $z^2+y^2=4$ and on the sides $x=0$ and $x=2$. Then value of the integral $\iiint_D x\,dx\,dy\,dz$ is

(a) $\pi-16$ (b) $\pi+16$ (c) π (d) $-\pi$

Soln: $I=\int\int\int_{x=0}^{2} x\,dx\,dy\,dz=2\int_{y=-2}^{2}\int_{z=2-y}^{\sqrt{4-y^2}} dz\,dy$

$$=2\int_{y=-2}^{2}\left(\sqrt{4-y^2}-2+y\right)dy=2\int_{-2}^{2}\sqrt{4-y^2}\,dy-4\int_{-2}^{2}dy+2\int_{-2}^{2}y\,dy$$

$$=2\times 2\int_0^2\sqrt{4-y^2}\,dy-4\times 4+2\times 0=4\times\frac{\pi}{4}-16 \quad \text{(hint : put } y=2\sin\theta\text{)}$$

$$=\pi-16$$

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Correct option is (a)

1.8 Vector Related Theorems

Divergence Theorem:

This theorem is applicable only for closed surfaces and this theorem converts surface integral into volume integral and vice versa.

If V is the volume bounded by a closed surface S and $\vec{A}$ is a vector function of position with continuous derivatives, then.

$$\oiint_S \vec{A}\cdot\overline{ds}=\oiint_S \vec{A}\cdot\hat{n}ds=\iiint_V(\vec{\nabla}\cdot\vec{A})dV$$

where $\hat{n}$ is the outward normal to 'S' indicating the positive direction of S.

Stokes Theorem:

This theorem is applicable only for open surfaces and this theorem converts surface integral into line integral and vice versa.

If S is an open, two-sided surface bounded by a closed, non-intersecting curve C, and $\vec{A}$ is a vector function of position with continuous derivatives; then

$$\oint_C \vec{A}\cdot\overrightarrow{dr} = \iint_S (\vec{\nabla}\times\vec{A}).\hat{n}ds = \iint_S (\vec{\nabla}\times\vec{A})\cdot\overrightarrow{ds}$$

where C is traversed in the positive (counterclockwise) direction.

Example 63: Calculate the flux of the vector field $\vec{V} = r^2\vec{r}$ through the surface S of a sphere of radius 'a', centered at the origin.

Soln: Flux of the vector field $\vec{V} = \oiint_S \vec{V}.d\vec{s} = \iiint_V (\vec{\nabla}.\vec{V})dv$ (Using divergence theorem)

$$= \iiint_V (\vec{\nabla}.r^2\vec{r})dv = \iiint_V 5r^2dv = 5\iiint_V r^2(r^2\sin\theta dr d\theta d\phi) = 5\int_0^a r^4dr\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi = 4\pi a^5$$

Example 64: Calculate $\oint_C \vec{A}.d\vec{r}$ for the vector field $\vec{A} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ along the boundary of upper half of the sphere $x^2+y^2+z^2 = 1$.

Soln: The boundary is the circle of unit radius in the x-y plane i.e. $x^2+y^2=1$. Therefore, substituting $x=\cos\theta$, $y=\sin\theta$ and $z=0$ we get,

$$\oint_C \vec{A}.d\vec{r} = \oint_C \left[(2x-y)dx - yz^2dy - y^2zdz\right] = \oint_C (2x-y)dx = \int_0^{2\pi}(2\cos\theta-\sin\theta)(-\sin\theta d\theta) = \pi$$

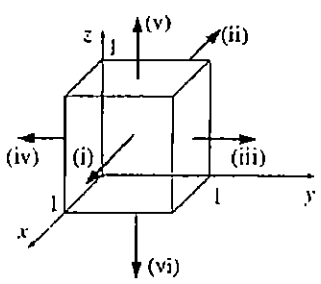
Example 65: Check the divergence theorem using the function

$$\vec{A} = y^2\hat{x} + (2xy+z^2)\hat{y} + (2yz)\hat{z}$$

over the unit cube situated at the origin shown in figure.

Soln: $\iiint_V (\vec{\nabla}.\vec{A})dv = \iiint_V 2(x+y)dv = 2\int_0^1\int_0^1\int_0^1 (x+y)dx\,dy\,dz$

$$= 2\left[\int_0^1\int_0^1\int_0^1 x\,dx\,dy\,dz + \int_0^1\int_0^1\int_0^1 y\,dx\,dy\,dz\right] = 2\left[\frac{1}{2}\times1\times1 + \frac{1}{2}\times1\times1\right] = 2$$



To evaluate the surface integral we must consider separately the six sides of the cube:

(i) $\int \vec{A}.d\vec{s} = \int_0^1\int_0^1 y^2\,dy\,dz = \frac{1}{3}$ (ii) $\int \vec{A}.d\vec{s} = -\int_0^1\int_0^1 y^2\,dy\,dz = -\frac{1}{3}$

(iii) $\int \vec{A}.d\vec{s} = -\int_0^1\int_0^1 (2x+z^2)\,dx\,dz = \frac{4}{3}$ (iv) $\int \vec{A}.d\vec{s} = -\int_0^1\int_0^1 z^2\,dx\,dz = -\frac{1}{3}$

(v) $\int \vec{A}.d\vec{s} = -\int_0^1\int_0^1 2y\,dx\,dy = 1$ (vi) $\int \vec{A}.d\vec{s} = -\int_0^1\int_0^1 0\,dx\,dy = 0$

So the total flux is:

$\oint_S \vec{A}.d\vec{s} = \frac{1}{3}-\frac{1}{3}+\frac{4}{3}-\frac{1}{3}+1+0 = 2$. Hence, divergence theorem is verified.

Example 66: Suppose $\vec{A} = (2xz+3y^2)\hat{j} + (4yz^2)\hat{k}$. Check Stokes' theorem for the square surface shown in figure.

Soln: Here, $\nabla\times\vec{A} = (4z^2-2x)\hat{i} + 2z\,\hat{k}$ and $d\vec{s} = dy\,dz\,\hat{i}$

Since x = 0 for this surface, $\int(\nabla\times\vec{A}).\vec{ds} = \int_0^1\int_0^1 4z^2\,dy\,dz = \frac{4}{3}$

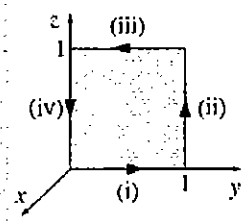
Now, for line integral, We must break this up into four segments:

(i) $x=0, z=0, \vec{A}.\vec{dl} = 3y^2dy \Rightarrow \int \vec{A}.\vec{dl} = \int_0^1 3y^2dy = 1$

(ii) $x=0, y=1, \vec{A}.\vec{dl} = 4z^2dz \Rightarrow \int \vec{A}.\vec{dl} = \int_0^1 4z^2dz = \frac{4}{3}$

(iii) $x=0, z=1, \vec{A}.\vec{dl} = 3y^2dy \Rightarrow \int \vec{A}.\vec{dl} = \int_1^0 3y^2dy = -1$

(iv) $x=0, y=0, \vec{A}.\vec{dl} = 0 \Rightarrow \int \vec{A}.\vec{dl} = \int_1^0 0dz = 0$



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So, $\oint \vec{A}.\vec{dl} = 1+\frac{4}{3}-1+0 = \frac{4}{3}$

Example 67: Evaluate $\oint_C (xy+y^2)dx + x^2dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

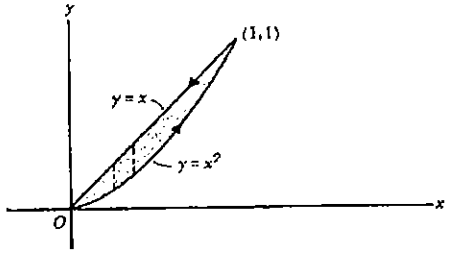
Soln: $\oint_C (xy+y^2)dx + x^2dy = \oint_C \left[(xy+y^2)\hat{i} + x^2\hat{j}\right].d\vec{r} = \iint_S \vec{\nabla}\times\left[(xy+y^2)\hat{i} + x^2\hat{j}\right].d\vec{S}$ (Stokes theorem)

For any region in x-y plane, $d\vec{S} = dxdy\,\hat{k}$

$= \iint_S (x-2y)\,dxdy = \int_{x=0}^1\int_{y=x^2}^x (x-2y)\,dydx$

$= \int_0^1\left[\int_{x^2}^x (x-2y)\,dy\right]dx = \int_0^1 (xy-y^2)\Big|_{x^2}^x\,dx$

$= \int_0^1 (x^4-x^3)\,dx = -\frac{1}{20}$



Example 68: Calculate the flux of the vector field $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ through the surface bounded by the region $x^2 + y^2 = 4,\ z = 0, z = 3$.

Soln: Flux of the vector field $= \iint_S \vec{F}.d\vec{s} = \iiint_V \left(\vec{\nabla}.\vec{F}\right) dv$ (according to divergence theorem)

$$= \iiint_V (4 - 4y + 2z)\, dxdydz = \int_{z=0}^{3}\int_{\phi=}^{2\pi}\int_{\rho=0}^{2} (4 - 4\rho\sin\phi + 2z)\, \rho d\rho d\phi dz = 84\pi$$

Example 69: Evalaute

$$\oint_C \left(x^2 - \cosh y\right) dx + \left(y + \sin x\right) dy$$

where C is the rectangle with vertices $(0,0), (\pi, 0), (\pi, 1), (0,1)$

Soln: $\oint_C \left(x^2 - \cosh y\right) dx + \left(y + \sin x\right) dy = \oint_C \left[\left(x^2 - \cosh y\right)\hat{i} + \left(y + \sin x\right)\hat{j}\right].d\vec{r}$

$$= \iint_S \vec{\nabla} \times \left[\left(x^2 - \cosh y\right)\hat{i} + \left(y + \sin x\right)\hat{j}\right].d\vec{S} \text{ (using Stokes theorem)}$$

For any region in x-y plane, $d\vec{S} = dxdy\ \hat{k}$

$$= \iint_S (\cos x + \sinh y)\, dxdy = \int_{x=0}^{\pi}\int_{y=0}^{1} (\cos x + \sinh y)\, dxdy$$

$$= \int_{x=0}^{\pi} \left[y\cos x + \cosh y\right]_{y=0}^{1} dx = \int_{x=0}^{\pi} \left[\cos x + \cosh - 1\right] dx = \left[\sin x + x\cosh 1 - x\right]_0^{\pi} = \pi\left(\cosh 1 - 1\right)$$

Example 70: Evaluate, $I = \iint_S \left(x^3 dydz + x^2 ydzdx + x^2 z\, dx\, dy\right)$

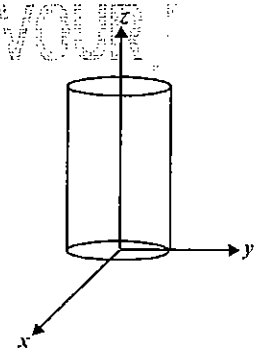
where S is the closed surface bounded by the planes $z = 0, z = b$ and the cylinder $x^2 + y^2 = a^2$.

Soln: By divergence theorem, the required surface integral I is equal to the volume integral.

$$\iiint_V \left[\frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(x^2 y\right) + \frac{\partial}{\partial z}\left(x^2 z\right)\right] dV$$

$$= \iiint 5x^2 dxdydz = 5\iiint \rho^2\cos^2\phi\, (\rho d\rho d\phi dz)$$

$$= 5\int_{\rho=0}^{a} \rho^3 d\rho \int_{\phi=0}^{2\pi} \cos^2\phi d\phi \int_{z=0}^{b} dz = 5.\frac{a^4}{4}.\pi b = \frac{5}{4}\pi b a^4$$



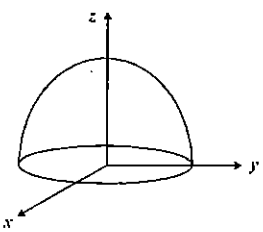
Example 71: Evaluate, $\iint_S \left(y^2 z^2\hat{i} + z^2 x^2\hat{j} + z^2 y^2\hat{k}\right).\hat{n}\, ds$

Where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane are bounded by this plane.

Soln: By divergence theorem, we have

$$\iint_S \left(y^2 z^2\hat{i} + z^2 x^2\hat{j} + z^2 y^2\hat{k}\right).\hat{n} ds = \iiint_V div\left(y^2 z^2\hat{i} + z^2 x^2\hat{j} + z^2 y^2\hat{k}\right) dV$$

$$= \iiint_V 2zy^2\, dV = 2\iiint_V zy^2 dV$$



Using spherical polar coordinates (r, θ, ϕ), the integral will be

$$= 2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(r\cos\theta)\left(r^2\sin^2\theta\sin^2\phi\right)r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^5\sin^3\theta\cos\theta\sin^2\phi\, dr\, d\theta\, d\phi = \frac{\pi}{2}$$

Example 72: If $F = \left(x^2+y-4\right)\hat{i} + 3xy\hat{j} + \left(2xz+z^2\right)\hat{k}$, evaluate $\iint_S \left(\nabla\times\vec{F}\right).\hat{n}dS$ where S is the surface of the sphere $x^2+y^2+z^2=16$ above the xy-plane.

Soln: S is the hemispherical surafce $x^2+y^2+z^2=16$ bounded by the circle C given by $x^2+y^2=16, z=0$.
Using Stokes theorem,

$$\iint_S \left(\vec{\nabla}\times\vec{F}\right).\hat{n}dS = \oint_C \vec{F}.d\vec{r} = \oint_C \left(x^2+y-4\right)dx + 3xydy$$

Putting $x = 4\cos\theta$ and $y = 4\sin\theta$, we get

$$= \int_{\theta=0}^{2\pi} \left(16\cos^2\theta + 4\sin\theta - 4\right)\left(-4\sin\theta d\theta\right) + 3\left(4\cos\theta\right)\left(4\sin\theta\right)\left(4\cos d\theta\right) = -16\pi$$

Example 73: Evaluate by Stokes theorem

$$\oint_C \left(e^x dx + 2ydy - dz\right)$$

where C is the curve $x^2+y^2=4, z=2$

Soln: $\oint_C \left(e^x dx + 2y\, dy - dz\right) = \oint_C \left(e^x\hat{i} + 2y\hat{j} - \hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = \oint_C \vec{F}.d\vec{r}$, where $F = e^x\hat{i} + 2y\hat{j} - \hat{k}$

Also, $curl\ F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = 0$

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Hence, $\oint_C \vec{F}.d\vec{r} = \iint_S \left(curl\ \vec{F}\right).\hat{n}\, dS = 0$, since curl F=0

Example 74: If $\vec{A} = \dfrac{-y\hat{x} + x\hat{y}}{y^2 + \frac{x^2}{4}}$, then the value of $\iint_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S}$, where S is the surface of paraboloid $z = 4-(x^2+4y^2)$ centered at origin, is

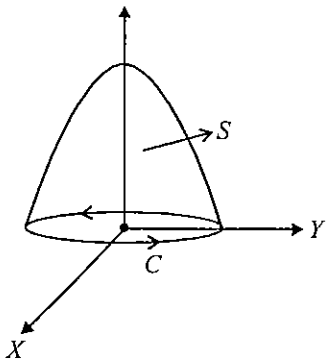
(a) 0 (b) 4π (c) -2π (d) π

Soln: If you find $\vec{\nabla}\times\vec{A}$, it is zero. Hence, $\iint_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S} = 0$

But it is wrong because given vector field $\vec{A}$ is not defined at the point (0, 0, 4) on the surface. Hence curl of vector field $\vec{A}$ will not be equal to zero at (0, 0, 4).

Given surface S is bounded by the curve C on x-y plane and

$C: x^2 + 4y^4 = 4$ (taken anti-clockwise)



According to Stoke's theorem,

$$\iint_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S} = \oint_C \vec{A}\cdot d\vec{r} = \oint_C \left(\frac{-ydx}{\frac{x^2}{4}+y^2} + \frac{xdy}{\frac{x^2}{4}+y^2}\right)$$

Put $x = 2\cos\theta \Rightarrow dx = -2\sin\theta d\theta$ and $y = \sin\theta \Rightarrow dy = \cos\theta d\theta$

$$\therefore \quad \oint_C \vec{A}\cdot d\vec{r} = \int_{\theta=0}^{2\pi} \frac{-\sin\theta\times -2\sin\theta d\theta + 2\cos\theta\times\cos\theta d\theta}{\frac{4\cos^2\theta}{4} + \sin^2\theta}$$

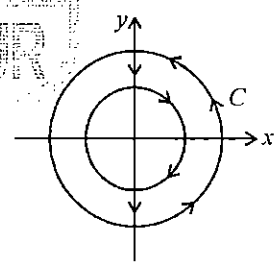
$$= \int_{\theta=0}^{2\pi} \left(2\sin^2\theta + 2\cos^2\theta\right) d\theta = 2\times 2\pi = 4\pi$$

Correct option is (b)

Example 75: The value of $\oint_C \vec{F}\cdot d\vec{r}$, where C is the curve bounded by $x^2 + y^2 \geq 4$; $x^2 + y^2 \leq 16$; $x \geq 0$ and $\vec{F} = -y\hat{i} + x\hat{j} + z\hat{k}$ is _____ $\times \pi$.

Soln: Using Stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \left(\vec{\nabla}\times\vec{F}\right)\cdot d\vec{S}$$



$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(1+1) = 2\hat{k}$$

and $d\vec{S} = dxdy\hat{k}$

$$\therefore \quad \iint_S \left(\vec{\nabla}\times\vec{F}\right)\cdot d\vec{S} = 2\iint dxdy$$

Put, $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rdrd\theta$

$$= 2\int_{\theta=-\pi}^{\pi}\int_{r=2}^{4} rdrd\theta = 2\left(\frac{r^2}{2}\right)_2^4 (2\pi) = 2\pi\times 12 = 24\,\pi$$

Correct answer is (24)

Example 76: If S is the closed surface enclosing a volume V and $\hat{n}$ is the unit normal vector to the surface and $\vec{r}$ is the positive vector, then the value of the folllowing integral $\iint_S \vec{r}\cdot\hat{n}dS$ is

(a) V (b) 2V (c) 0 (d) 3V

Soln: $\iint_S \vec{r}\cdot\hat{n}dS = \iiint_V \left(\vec{\nabla}\cdot\vec{r}\right)dV$ (using Divergence Theorem)

$= 3\iiint_V dV = 3V$

Correct option is (d)

Example 77: The value of $\oint_S \frac{\vec{r}.d\vec{s}}{r^3}$, where $\vec{r}$ is the position vector ans S is the closed surface enclosing the origin, is

(a) zero (b) π (c) 4π (d) 8π

Soln: $\oint_s \frac{\vec{r}\cdot d\vec{s}}{r^3} = \int_V \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)dV$ (using divergence theorem)

Since the closed surface encloses the origin i.e. $\vec{r}=0$

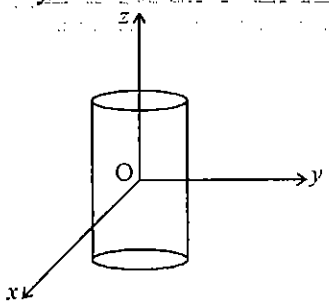
Therefore, $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = 4\pi\,\delta^3\left(\vec{r}\right)$

So, $\int_V \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)dV = 4\pi\int_V \delta^3\left(\vec{r}\right)dV = 4\pi$

Correct option is (c)

Example 78: Consider a cylinder of height h and radius a, closed at both ends, centered at the origin. Let $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ be the position vector and $\hat{n}$ a unit vector normal to the surface. The surface integral $\int_S \vec{r}\cdot\hat{n}\,dS$ over the closed surface of the cylinder is :

CAREER ENDEAVOUR



(a) $2\pi a^2(a+h)$ (b) $3\pi a^2 h$ (c) $2\pi a^2 h$ (d) zero

Soln: $\int_S \vec{r}\cdot\hat{n}\,dS = \int_V (\vec{\nabla}\cdot\vec{r})\,dV$ (Using divergence theorem)

$= 3\int_V dV = 3V = 3\pi a^2 h$

1.9 Linear Dependency of Vectors

Three vectors $\bar{u}, \bar{v}, \bar{w}$ will be linearly dependent if $\alpha_1\bar{u} + \alpha_2\bar{v} + \alpha_3\bar{w} = 0$, where α_1, α_2, α_3 are not all zero. These three independent vectors do not form the basis of $\mathbb{R}^3$ space.

If $\alpha_1\bar{u} + \alpha_2\bar{v} + \alpha_3\bar{w} = 0$ such that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then $\bar{u}, \bar{v}, \bar{w}$ are said to be linearly independent and they do form the basis of $\mathbb{R}^3$ space i.e. any vector in $\mathbb{R}^3$ space can be written as the linear combination of vectors $\bar{u}, \bar{v}, w$ respectively.

How to check:

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \begin{cases} = 0 \text{ for linearly dependent vectors} \\ \neq 0 \text{ for linearly independent vectors} \end{cases}$$

Example 79: Consider the set of vectors $\frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{2}}(0,1,1)$ and $\frac{1}{\sqrt{2}}(1,0,1)$

(a) the three vectors are orthonormal
(b) the three vectors are linearly independent
(c) the three vectors cannot form a basis in a three-dimensional real vector space
(d) $\frac{1}{\sqrt{2}}(1,1,0)$ can be written as the linear combination of $\frac{1}{\sqrt{2}}(0,1,1)$ and $\frac{1}{\sqrt{2}}(1,0,1)$

Soln: $A = \frac{1}{\sqrt{2}}(1,1,0);\ B = \frac{1}{\sqrt{2}}(0,1,1);\ C = \frac{1}{\sqrt{2}}(1,0,1)$

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{\sqrt{2}}\left(\frac{1}{2} - 0\right) - 0\left(\frac{1}{2} - 0\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{2} - 0\right) = \frac{1}{\sqrt{2}} \neq 0$$

So, $\bar{A}, \bar{B}, \bar{C}$ are linearly independent.

Correct option is (b)

Example 80: Given the four vectors, $u_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} 2 \\ 4 \\ -8 \end{pmatrix}, u_4 = \begin{pmatrix} 3 \\ 6 \\ -12 \end{pmatrix}$, the linearly dependent pair is

(a) u_1, u_2 (b) u_1, u_3 (c) u_1, u_4 (d) u_3, u_4

Soln: $u_4 = \frac{3}{2}u_3$ i.e. u_3, u_4 are linearly dependent

Correct option is (d)

Example 81: Consider the set of vectors in three-dimensional real vector space R^3,

$$S=\{(1,1,1),(1,-1,1),(1,1,-1)\}$$

Which of the following statement is true?
(a) S is not a linearly independent set
(b) S is a basis for R^3
(c) The vectors in S are orthogonal
(d) An orthogonal set of vectors cannot be generated from S

Soln: $\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=1(1-1)-1(-1-1)+1(1+1)=4\neq 0$

Correct option is (b)

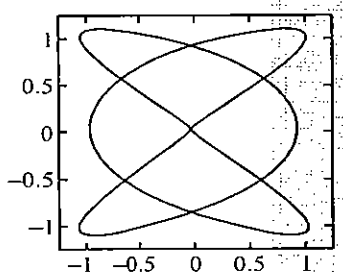
TIFR PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. A two-dimensional vector $\bar{A}(t)$ is given by $\bar{A}(t)=\hat{i}\sin 2t+\hat{j}\cos 3t$

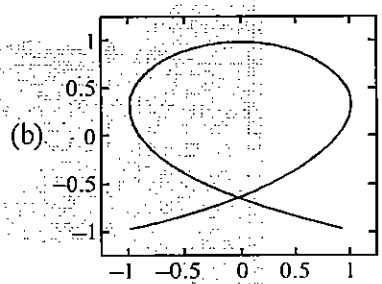
Which of the following graphs best describes the locus of the tip of the vector, as t is varied from 0 to 2π?

[TIFR 2013]

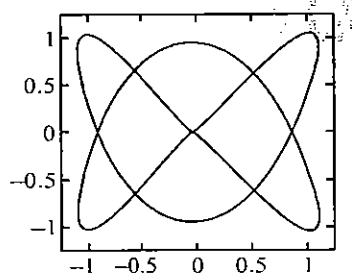
(a)



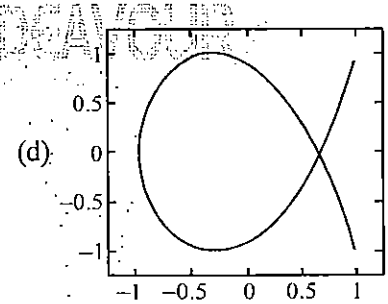
(b)



(c)



(d)



Soln: $\bar{A}(t)=\hat{i}\sin 2t+\hat{j}\cos 3t$

Here, $X(t)=\sin 2t$ and $Y(t)=\cos 3t$

Let us find the number of zeros along x-axis and y-axis.

Along y-axis

$X(t)=0\Rightarrow \sin 2t=0$

$2t=n\pi \;;\; n=0,\pm 1,\pm 2,....$

$n=0 \quad t=0$

Along x-axis

$Y(t)=0\Rightarrow \cos 3t=0$

$3t=(2n+1)\frac{\pi}{2} \;;\; n=0,\pm 1,\pm 2,....$

$n=0 \quad t=\frac{\pi}{6}$

$n=1 \quad t=\frac{\pi}{2}$

$n=2 \quad t=\pi$

$n=3 \quad t=\frac{3\pi}{2}$

$n=4 \quad t=2\pi$

$n=1 \quad t=\frac{\pi}{2}$

$n=2 \quad t=\frac{5\pi}{6}$

$n=3 \quad t=\frac{7\pi}{6}$

$n=4 \quad t=\frac{9\pi}{6}$

$n=5 \quad t=\frac{11\pi}{6}$

$n=6 \quad t=\frac{13\pi}{6}$

Hence, $X(t)=0$ for four times in one period (2π).

Hence, $Y(t)=0$ is zero for six times in one period (2π)

The curve given in option (c) passes thorugh $x=0$ four times and $y=0$ six times.

Correct option is (c)

Question 2. Consider the surface corresponding to the equation $4x^2+y^2+z=0$. A possible unit tangent to this surface at the point (1, 2, –8) is **[TIFR 2013]**

(a) $\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{j}$

(b) $\frac{1}{5}\hat{j}-\frac{4}{5}\hat{k}$

(c) $\frac{4}{9}\hat{i}-\frac{8}{9}\hat{j}+\frac{1}{9}\hat{k}$

(d) $-\frac{1}{\sqrt{5}}\hat{i}+\frac{3}{\sqrt{5}}\hat{j}-\frac{4}{\sqrt{5}}\hat{k}$

Soln: Given surface is $\phi(x, y, z)=4x^2+y^2+z=0$.

Hence normal to this surface is $\vec{\nabla}\phi=8x\hat{i}+2y\hat{j}+\hat{k}$. Therefore, $\vec{\nabla}\phi\big|_{(1,2,-8)}=8\hat{i}+4\hat{j}+\hat{k}$.

Now, the dot product of $\vec{\nabla}\phi$ with the tangent vector should be zero. Here, option (a), (b) and (d) satisfies the condition. But only the vector given in option (a) has unit magnitude.

Correct option is (a)

CAREER ENDEAVOUR

Question 3. Which of the following vectors is parallel to the surface $x^2y+2xz=4$ at the point (2,-2,3)? **[TIFR 2015]**

(a) $-6\hat{i}-2\hat{j}+5\hat{k}$ (b) $6\hat{i}+2\hat{j}+5\hat{k}$ (c) $6\hat{i}-2\hat{j}+5\hat{k}$ (d) $6\hat{i}-2\hat{j}-5\hat{k}$

Soln: Given surface is $\phi(x, y, z)=x^2y+2xz-4=0$.

Hence, normal to this surface is $\vec{\nabla}\phi=(2xy+2z)\hat{i}+x^2\hat{j}+2x\hat{k}$. Therefore, $\vec{\nabla}\phi\big|_{(2,-2,3)}=-2\hat{i}+4\hat{j}+4\hat{k}$

Now, the dot product of $\vec{\nabla}\phi$ with vector parallel to the surface, should be zero. Here, only option (c) satisfies the condition.

Correct option is (c)

JEST PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The vector field $xz\hat{i}+y\hat{j}$ in cylinderical polar coordinates is **[JEST 2013]**

(a) $\rho\left(z\cos^2\phi+\sin^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi\left(1-z\right)\hat{e}_\phi$

(b) $\rho\left(z\cos^2\phi+\sin^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi\left(1+z\right)\hat{e}_\phi$

(c) $\rho\left(z\sin^2\phi+\cos^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi\left(1+z\right)\hat{e}_\phi$

(d) $\rho\left(z\sin^2\phi+\cos^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi\left(1-z\right)\hat{e}_\phi$

Soln: We know that in cylindrical coordinates, $x=\rho\cos\phi,\ y=\rho\sin\phi,\ z=z$

Also, $\hat{\rho}=\cos\phi\,\hat{i}+\sin\phi\,\hat{j}$ ...(1)

and $\hat{\phi}=-\sin\phi\,\hat{i}+\cos\phi\,\hat{j}$ ...(2)

and $\hat{z}=\hat{k}$

Solving equation (1) & (2), we can get

$$\hat{i}=\cos\phi\,\hat{\rho}-\sin\phi\hat{\phi}\ ;\ \hat{j}=\sin\phi\,\hat{\rho}+\cos\phi\hat{\phi}$$

Therefore, $\left(xz\,\hat{i}+y\hat{j}\right)=z\rho\cos\phi\left(\cos\phi\hat{\rho}-\sin\phi\hat{\phi}\right)+\rho\sin\phi\left(\sin\phi\hat{\rho}+\cos\phi\hat{\phi}\right)$

$$=\rho\left(z\cos^2\phi+\sin^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi\left(1-z\right)\hat{e}_\phi$$

Correct option is (a)

Question 2. What is the equation of the plane which is tangent to the surface $xyz=4$ at the point (1, 2, 2) ? **[JEST 2017]**

(a) $x+2y+4z=12$ (b) $4x+2y+z=12$ (c) $x+4y+z=0$ (d) $2x+y+z=6$

Soln: Let $\phi: xyz-4=0$

$$\therefore\quad \frac{\partial\phi}{\partial x}=yz\Rightarrow\left.\frac{\partial\phi}{\partial x}\right|_{1,2,2}=4,\ \frac{\partial\phi}{\partial y}=xz\Rightarrow\left.\frac{\partial\phi}{\partial y}\right|_{1,2,2}=2,\ \frac{\partial\phi}{\partial z}=xy\Rightarrow\left.\frac{\partial\phi}{\partial y}\right|_{1,2,2}=2$$

Therefore, equation of tangent plane at the point (1, 2, 2)

$$(x-1)\times4+(y-2)\times2+(z-2)\times2=0\ \Rightarrow 4x+2y+2z=12\ \Rightarrow 2x+y+z=6$$

Correct option is (d)

JNU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The position vector of a particle moving in two dimensions, as a function of time t, is given as $\vec{r}(t)=\left(\hat{i}t+\hat{j}\sqrt{t}\right)e^{-t}$ for $t\geq0$. Starting at $t=0$, the distance of the particle from the origin reaches its maximum value at $t=t_m$, and at this instant, the velocity vector $\vec{v}$ makes an angle α with $\vec{r}$. The values of t_m and α are **[JNU 2014]**

(a) $t_m = \frac{1}{2}$ and $\alpha = \frac{\pi}{2}$ (b) $t_m = \frac{1}{\sqrt{2}}$ and $\alpha = \pi$

(c) $t_m = \frac{1}{2}$ and $\alpha = 0$ (d) $t_m = \frac{1}{\sqrt{2}}$ and $\alpha = \frac{\pi}{2}$

(e) $t_m = 1$ and $\alpha = \pi$

Soln: Given $\vec{r}(t) = \left(\hat{i}\, t + \hat{j}\sqrt{t}\right)e^{-t}\ ;\ t \geq 0$

Distance of particle from origin $= |\vec{r}(t)| \Rightarrow |\vec{r}(t)| = \sqrt{(t^2+t)e^{-2t}}$.

For maxima and minima, $\frac{d|\vec{r}(t)|}{dt} = 0 \quad \Rightarrow \frac{(2t+1)e^{-2t} + (t^2+t)\times -2e^{-2t}}{\sqrt{(t^2+t)e^{-2t}}} = 0$

$\Rightarrow (2t+1) - 2(t^2+t) = 0 \quad \Rightarrow -2t^2 + 1 = 0 \Rightarrow t = \frac{1}{\sqrt{2}}$

Therefore, $t_m = \frac{1}{\sqrt{2}}$.

Velocity of particle, $\vec{V}(t) = \frac{d\vec{r}(t)}{dt} = \hat{i}\left(e^{-t} - te^{-t}\right) + \hat{j}\left(\frac{1}{2\sqrt{t}}e^{-t} - \sqrt{t}\,e^{-t}\right) = \left\{\hat{i}(1-t) + \hat{j}\left(\frac{1}{2\sqrt{t}} - \sqrt{t}\right)\right\}e^{-t}$

Here, $\vec{V}\cdot\vec{r} = \left[\left\{\hat{i}(1-t) + \hat{j}\left(\frac{1}{2\sqrt{t}} - \sqrt{t}\right)\right\}e^{-t}\right]\cdot\left[\left(\hat{i}\,t + \hat{j}\sqrt{t}\right)e^{-t}\right] = e^{-2t}\left[t(1-t) + \left(\frac{1}{2} - t\right)\right]$

If at $t = t_m$, the velocity vector $\vec{v}$ makes an angle α with $\vec{r}$, then

$$\cos\alpha = \left.\frac{\vec{V}\cdot\vec{r}}{|\vec{V}||\vec{r}|}\right|_{t=t_m=\frac{1}{\sqrt{2}}} = 0 \Rightarrow \alpha = \frac{\pi}{2}$$

UNACADEMY THE FUTURE IS HERE

Correct option is (d)

Question 2. Consider a vector field $\vec{F} = xy^4\hat{i} + yz^2\hat{j}$. Its outward flux $\oint \vec{F}\cdot dS$ over the surface of a cube bounded by the planes $|x| = 2$, $|y| = 2$ and $|z| = 2$ is nearest to **[JNU 2014]**

(a) 410 (b) –273 (c) 290 (d) –300 (e) 0

Soln: According to Divergence Theorem,

$$\oint_S \vec{F}.ds = \int_V \left(\vec{\nabla}.\vec{F}\right)dV = \int_{z=-2}^{2}\int_{y=-2}^{2}\int_{x=-2}^{2}\left(y^4 + z^2\right)dxdydz$$

$$= x\Big|_{-2}^{2}\frac{y^5}{5}\Big|_{-2}^{2}z\Big|_{-2}^{2} + x\Big|_{-2}^{2}y\Big|_{-2}^{2}\frac{z^3}{3}\Big|_{-2}^{2} = \left(4\times\frac{64}{5}\times 4\right) + \left(4\times 4\times\frac{16}{3}\right) = 291.13$$

Correct option is (c)

Question 3. Which of the following vector field can represents an electrostatic field? **[JNU 2015]**

(a) $\left(2xz-y^2\right)\hat{i}+\left(2xy-z^2\right)\hat{j}+\left(2yz-x^2\right)\hat{k}$ (b) $xy\hat{i}+2yz\hat{j}+3xz\hat{k}$

(c) $yz\hat{i}-xz\hat{j}+xy\hat{k}$ (d) $y^2\hat{i}+\left(2xy+z^2\right)\hat{j}+\left(2yz+x^2\right)\hat{k}$ (e) $y^2\hat{i}+\left(2xy+z^2\right)\hat{j}+2yz\hat{k}$

Soln: Since, curl of electrostatic field is zero, therefore check curl of all options.

For option (c) ; $\vec{\nabla}\times\vec{E}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ yz & xz & xy\end{vmatrix}=\hat{i}(x-x)-\hat{j}(y-y)+\hat{k}(z-z)=0$

Correct option is (c)

Question 4. Define I to be the value of the surface integral ($\int \mathbf{E}\cdot d\mathbf{S}$, where $d\mathbf{S}$ points outwards from the domain on integration) of a vector field $\mathbf{E}(\mathbf{E}=(x+y^2)\mathbf{i}+(z^3+y^3)\mathbf{j}+(x+z^4)\mathbf{k})$ over the entire surface of a cube which bounds the region $\{0<x<2,\ -1<y<1,\ 0<z<2\}$. The value of I is **[JNU 2016]**

(a) 0 (b) 16 (c) 72 (d) 80 (e) 32

Soln: According to Divergence Theorem,

$$\int_S \vec{E}.d\vec{s}=\int_V\left(\vec{\nabla}.\vec{E}\right)dV=\int_{z=0}^{2}\int_{y=-1}^{1}\int_{x=0}^{2}\left(1+3y^2+4z^3\right)dxdydz$$

$$=x\Big|_0^2 y\Big|_{-1}^1 z\Big|_0^2+x\Big|_0^2 y^3\Big|_{-1}^1 z\Big|_0^2+x\Big|_0^2 y\Big|_{-1}^1 z^4\Big|_0^2$$

$$=(2\times2\times2)+(2\times2\times2)+(2\times2\times16)=80$$

Correct option is (d)

Question 5. Which of the following force fields is/are conservative ? **[JNU 2016]**

I. $(4x^3y+3x^2z^2)\mathbf{i}+(x^4+3y^2z^2)\mathbf{j}+2z(x^3+y^3)\mathbf{k}$

II. $(yz^4+x^5)\mathbf{i}+(x^4z+x^2y^3)\mathbf{j}+(x^4y+z^5)\mathbf{k}$

III. $x(3xy^2+4z^2)\mathbf{i}+(z^4+2x^3y)\mathbf{j}+2z^2(2yz+3x^2)\mathbf{k}$

(a) II and III only (b) I, II and III (c) I and III only (d) I and II only (e) I only

Soln:

Question 6. The line integral $I=\int_\Gamma \vec{F}\cdot d\vec{r}$ of the vector field $\vec{F}=yz\,i+zx\,j+xy\,k$ is evaluated along the curve Γ parametrized by $(x=a\sin t,\ y=b\cos t,\ z=c)$, where a, b and c and c are constant. The value of I for $\frac{\pi}{4}\le t\le\frac{3\pi}{4}$ is **[JNU 2017]**

(a) abc (b) ab (c) bc (d) ac (e) 0

Soln: $I=\int_\Gamma \vec{F}\cdot d\vec{r}=\int_\Gamma(yz\,dx+zx\,dy+xy\,dz)$

Put $x=a\sin t\ \Rightarrow dx=a\cos t\,dt,\ y=b\cos t\ \Rightarrow dy=-b\sin t dt,\ z=c\ \Rightarrow dz=0$

$$\therefore \quad I = \int_{t=\pi/4}^{3\pi/4} b\cos t \times c \cdot a\cos t\, dt + c \cdot a\sin t \times -b\sin t\, dt + 0$$

$$= \int_{\pi/4}^{3\pi/4} abc(\cos^2 t - \sin^2 t)\, dt$$

$$= abc \int_{\pi/4}^{3\pi/4} \cos 2t\, dt = abc\left[\frac{\sin 2t}{2}\right]_{\pi/4}^{3\pi/4} = \frac{abc}{2}[-1-1] = -abc$$

Correct option is (a)

DU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. A vector field is given by $\vec{A} = y\hat{i} + z\hat{j} + x\hat{k}$. The curl of the vector is: **[DU 2014]**

(a) $\hat{i}+\hat{j}-\hat{k}$ (b) $-\hat{i}-\hat{j}-\hat{k}$ (c) $-\hat{i}-\hat{j}+\hat{k}$ (d) $\hat{i}+\hat{j}+\hat{k}$

Soln: $\vec{A} = y\hat{i} + z\hat{j} + x\hat{k}$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = \hat{i}(0-1) - \hat{j}(1-0) + \hat{k}(0-1) = -\hat{i} - \hat{j} - \hat{k}.$$

Correct option is (b)

Question 2. The magnitude of the cross product of two vectors $\overline{AB}$ and $\overline{AC}$ is equal to: **[DU 2014]**

(a) The area of the triangle
(b) The area of the parallelogram with adjacent sides AB and AC
(c) Twice the area of the rectangle with adjacent sides AB and AC
(d) The area of the circle passing through A, B and C

Soln: Correct option is (b)

CAREER ENDEAVOUR

Question 3. Consider a set of surfaces $f(x, y, z) = c$. Different values of c will generate different surfaces. The gradient of f at a point P has a direction: **[DU 2014]**

(a) Tangential to the surface at point P
(b) At 45 degrees to the surface at point P
(c) Normal to the surface at the point P in the direction of increasing value of f
(d) Normal to the surface at the point P in the direction of decreasing value of f

Soln: Correct option is (c)

Question 4. Let $\vec{r} = \hat{x}r\cos\omega t + \hat{y}r\sin\omega t$. For constant r and ω, calculate $\vec{r} \times \frac{d\vec{r}}{dt}$. **[DU 2015]**

(a) $r^2\omega\hat{x}$ (b) $r^2\omega\hat{y}$ (c) $r^2\omega\hat{z}$ (d) $r^2\omega\cos(2\omega t)\hat{z}$

Soln: $\vec{r} = \hat{x}r\cos\omega t + \hat{y}r\sin\omega t$

$$\frac{d\vec{r}}{dt} = \hat{x}r \times -\omega\sin\omega t + \hat{y}r\omega\cos\omega t$$

Therefore, $\vec{r}\times\frac{d\vec{r}}{dt}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ r\cos\omega t & r\sin\omega t & 0\\ -r\omega\sin\omega t & r\omega\cos\omega t & 0\end{vmatrix}\hat{z}\left(r^2\omega\cos^2\omega t+r^2\omega\sin^2\omega t\right)$

$=\hat{z}\,r^2\omega$

Correct option is (c)

Question 5. The magnitude of $\oint\vec{r}\times\overline{dr}$ over the perimeter of an ellipse described by

$\vec{r}=\hat{x}a\cos\theta+\hat{y}b\sin\theta$, is **[DU 2015]**

(a) $\pi ab/2$ (b) πab (c) $2\pi ab$ (d) ab

Soln: $\vec{r}=\hat{x}\,a\cos\theta+\hat{y}\,b\sin\theta$ and $d\vec{r}=\hat{x}\,a\times-\sin\theta\,d\theta+\hat{y}\,b\cos\theta\,d\theta$

$$\vec{r}\times d\vec{r}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ a\cos\theta & b\sin\theta & 0\\ -a\sin\theta\,d\theta & b\cos\theta\,d\theta & 0\end{vmatrix}=\hat{k}\left(ab\cos^2\theta+ab\sin^2\theta\right)=\hat{k}\,ab\,d\theta$$

Therefore, $\oint\vec{r}\times d\vec{r}=\hat{k}\,ab\oint d\theta=2\pi\,ab\,\hat{k}$.

Correct option is (c)

Question 6. Which of the following sets of vectors constitute a basis for the plane? **[DU 2016]**

(i) (0, 1), (1, 1) (ii) (1, 1), (1, −1) (iii) (1, 0), (0, −1)

(a) all three sets of vector (b) only (iii)

(c) only (i) and (iii) (d) only (ii) and (iii)

Soln: Set (i): Vectors (0,1) and (1,1) cannot be written as a scalar multiple of each other.
Set (ii): Vectors (1,1) and (1,-1) cannot be written as a scalar multiple of each other.
Set (iii): Vectors (1,0) and (0,-1) cannot be written as a scalar multiple of each other.
Therefore, all sets of vectors are sets of linearly independent vectors and they will constitute a basis for the plane.

Correct option is (a)

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Question 7. Given the position vector $\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}$ and $r^2=\vec{r}\cdot\vec{r}$, the value of $\vec{\nabla}\cdot(r^2\vec{r})$ is given by:

(a) $6r^2$ (b) $5r^2$ (c) $4r^2$ (d) $3r^2$ **[DU 2017]**

Soln: $\vec{\nabla}.\left(r^2\vec{r}\right)=\vec{\nabla}.\left(r^3\hat{r}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^5\right)=5r^2$

Correct option is (b)

HCU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. If $\vec{r}$ is a position vector, then $\vec{\nabla} \times \vec{r}$ is given by

(a) $r^{3/2}$ (b) $r^{1/2}$ (c) 0 (d) 3 **[H.C.U. 2010]**

Soln: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0$$

Correct option is (c)

Question 2. For a given vector $\vec{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$, the value of the integral $\iint_S \vec{A} \cdot d\vec{S}$, over the cube defined by $0 \le x, y, z \le 1$ is given by

(a) 0 (b) 2 (c) 3 (d) 4 **[H.C.U. 2010]**

Soln: According to Divergence theorem,

$$\iint_S \vec{A} \cdot d\vec{S} = \iiint_V \left(\vec{\nabla} \cdot \vec{A}\right) dV = \int_{z=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (2x + 2y + 2z)\, dx\, dy\, dz$$

$$= x^2\Big|_0^1 y\Big|_0^1 z\Big|_0^1 + x\Big|_0^1 y^2\Big|_0^1 z\Big|_0^1 + x\Big|_0^1 y\Big|_0^1 z^2\Big|_0^1 = 1+1+1 = 3$$

Correct option is (c)

Question 3. Which of the following can represent a conservative force? ($\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors)

[H.C.U. 2011]

(a) $C(\hat{i}y + \hat{j}z + \hat{k}z)$ (b) $C(\hat{i}x + \hat{j}y + \hat{k}y)$ (c) $C \cdot xyz(\hat{i} + \hat{j} + \hat{k})$ (d) $C(\hat{i}x + \hat{j}y + \hat{k}z)$

Soln: For field $\vec{A}$ to be conservative, $\vec{\nabla} \times \vec{A} = 0$. Only option (d) satifies the condition.

Correct option is (d)

Question 4. What should be the value of a such that the straight line $x + y = a$ will be a tangent to the circle $x^2 + y^2 = 4$?

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) 4 **[H.C.U. 2011]**

Soln: For $x + y = a$ to be tangent to the circle, $x^2 + y^2 = 4$,

Perpendicular distance to the line $x + y = a$ from the center of the circle (0,0) will be equal to radius of the circle i.e. $\left|\frac{x + y - a}{\sqrt{1+1}}\right|_{(0,0)} = 2 \Rightarrow \left|\frac{a}{\sqrt{2}}\right| = 2 \Rightarrow a = 2\sqrt{2}$

Correct option is (c)

Question 5. $\nabla^2(\ln r)$ is equal to **[H.C.U.-2011]**

(a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$ (c) $\frac{1}{r^3}$ (d) $\frac{1}{r}+\frac{1}{r^2}$

Soln: $\nabla^2(\ln r)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\ln r\right)$

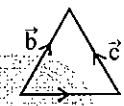
$=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\times\frac{1}{r}\right)=\frac{1}{r^2}.$

Correct option is (b)

Question 6. If two sides of triangle are represented by $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=2\hat{j}+3\hat{k}$, then the third side is represented by **[H.C.U. 2012]**

(a) $\hat{i}-3\hat{j}+2\hat{k}$ (b) $\hat{i}+3\hat{j}+2\hat{k}$ (c) $-\hat{i}+\hat{j}-2\hat{k}$ (d) $-\hat{i}+\hat{j}+2\hat{k}$

Soln: In a triangle we can write, $\vec{a}+\vec{c}=\vec{b}\Rightarrow\vec{c}=\vec{b}-\vec{a}=-\hat{i}+\hat{j}+2\hat{k}$



Correct option is (d)

Question 7. A unit vector normal to the surface $z=x^2+y^2$ at the point (1, 2, 5) is given by **[H.C.U. 2012]**

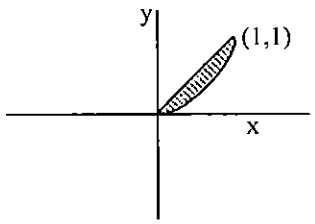
(a) $\pm\frac{\hat{i}+2\hat{j}-\hat{k}}{\sqrt{6}}$ (b) $\pm\frac{2\hat{i}+2\hat{j}-2\hat{k}}{\sqrt{12}}$ (c) $\pm\frac{2\hat{i}+4\hat{j}-\hat{k}}{\sqrt{21}}$ (d) $\pm\frac{2\hat{i}+4\hat{j}-2\hat{k}}{\sqrt{24}}$

Soln: Unit normal to the surface $=\left.\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}\right|_{(1,2,5)}=\left.\frac{2x\,\hat{i}+2y\,\hat{j}-\hat{k}}{\sqrt{4x^2+4y^2+1}}\right|_{(1,2,5)}=\pm\frac{2\hat{i}+4\hat{j}-\hat{k}}{\sqrt{21}}.$

Correct option is (c)

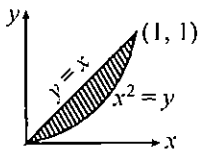
CAREER ENDEAVOUR

Question 8. The area of the shaded region between the straight line and parabola as shown in the figure is given by **[H.C.U. 2012]**



(a) $\int_0^1 dy\int_{x^2}^{x} dx$ (b) $\int_0^1 dx\int_{x^2}^{x} dy$ (c) $\int_0^1 dx\int_0^{x} dy$ (d) $\int_0^1 dy\int_0^{x^2} dx$

Soln: Area $=\iint dx\,dy=\int_0^1 dy\int_{x=y}^{x=\sqrt{y}} dx=\int_0^1 dx\int_{x^2}^{x} dy$



Correct option is (b)

Question 9. The number of vectors orthogonal to $\vec{i}+2\vec{j}-\vec{k}$ is **[H.C.U. 2013]**

(a) One (b) Two (c) Three (d) Infinite

Soln: There can be many vectors orthogonal to a given vector.

Correct option is (d)

Question 10. $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=0$ implies that **[H.C.U. 2013]**

(a) All the three vectors are parallel.
(b) All the three vectors are mutually perpendicular.
(c) Two of the three vectors are perpendicular
(d) The vectors lie in a plane.

Soln: $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=0$

We know that $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)$ gives the volume of a parallelopiped whose adjacent sides are $\vec{A}$, $\vec{B}$ and $\vec{C}$.

Therefore, $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=0$ means all vectors lie in a plane.

Correct option is (d)

Question 11. For a unit sphere, with center at the origin the equation of the tangent plane at the point $\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ is **[H.C.U. 2013]**

(a) $x+y+z=\frac{1}{\sqrt{3}}$ (b) $x-y+z=\frac{1}{\sqrt{3}}$ (c) $x-y-z=\frac{1}{\sqrt{3}}$ (d) $x+y-z=\frac{1}{\sqrt{3}}$

Soln: Equation of unit sphere $\phi: x^2+y^2+z^2-1=0$

Equation of tangent plane at the point (x_1, y_1, z_1), will be

$$(x-x_1)\left.\frac{\partial\phi}{\partial x}\right|_{(x,y,z)}+(y-y_1)\left.\frac{\partial\phi}{\partial y}\right|_{(x,y,z)}+(z-z_1)\left.\frac{\partial\phi}{\partial z}\right|_{(x,y,z)}=0$$

$\Rightarrow$ Here, $\frac{\partial\phi}{\partial x}=2x \Rightarrow \left.\frac{\partial\phi}{\partial x}\right|_{\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)}=\frac{2}{\sqrt{3}}$, $\quad \frac{\partial\phi}{\partial y}=2y \Rightarrow \left.\frac{\partial\phi}{\partial y}\right|_{\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)}=-\frac{2}{\sqrt{3}}$

and $\frac{\partial\phi}{\partial z}=2z \Rightarrow \left.\frac{\partial\phi}{\partial z}\right|_{\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)}=\frac{2}{\sqrt{3}}$

Therefore, equation of tangent plane to the given surface will be

$$\left(x-\frac{1}{\sqrt{3}}\right)\frac{2}{\sqrt{3}}+\left(y+\frac{1}{\sqrt{3}}\right)\left(-\frac{2}{\sqrt{3}}\right)+\left(z-\frac{1}{\sqrt{3}}\right)\frac{2}{\sqrt{3}}=0 \Rightarrow x-y+z=\frac{1}{\sqrt{3}}.$$

Correct option is (d)

Question 12. The divergence of $\vec{c}\times(\vec{r}\times\vec{c})$ where $\vec{c}$ is a numerical vector is given by

(a) zero (b) $|\vec{c}|^2$ (c) $2|\vec{c}|^2$ (d) $-2|\vec{c}|^2$ **[H.C.U. 2013]**

Soln: $\vec{c}\times(\vec{r}\times\vec{c})=\vec{r}(\vec{c}\cdot\vec{c})-\vec{c}(\vec{c}\cdot\vec{r})$

$$\therefore\ \vec{\nabla}\cdot\left[\vec{c}\times(\vec{r}\times\vec{c})\right]=c^2\,\vec{\nabla}\cdot\vec{r}-\vec{\nabla}.\left[\vec{c}(\vec{c}\cdot\vec{r})\right]$$
$$=3c^2-\left[\vec{\nabla}(\vec{c}\cdot\vec{r}).\vec{c}+(\vec{c}\cdot\vec{r})(\vec{\nabla}.\vec{c})\right]$$
$$=3c^2-[\vec{c}.\vec{c}+0]=2c^2$$

Correct option is (c)

Question 13. The value of the line integral $\int(y^2+xy)\,dx+(x^2-xy)\,dy$ taken along the straight line joining (0, 0) to (2, 1) is **[H.C.U. 2013]**

(a) 64/3 (b) 32/3 (c) 16/3 (d) 8/3

Soln: Equation of straight line joining $A(0,0)$ and $B(2,1)$ is,

$$y-0=\frac{1-0}{2-0}(x-0)\Rightarrow\boxed{2y=x}\Rightarrow dx=2dy$$

$$\therefore\ \int(y^2+xy)\,dx+(x^2-xy)\,dy=\int_0^1\left[(y^2+2y.y)2dy+(4y^2-2y.y)\,dy\right]$$
$$=\left(\frac{6y^3}{3}+\frac{2y^3}{3}\right)_0^1=2+\frac{2}{3}=\frac{8}{3}.$$

Correct option is (d)

Question 14. The value of the integral, $\int_0^1 dx\int_0^x dy\,xy$ is **[H.C.U. 2014]**

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{2}$

Soln: $\int_0^1 dx\int_0^x dy\,xy=\int_0^1 dx\int_0^x xy\,dy=\int_0^1 dx\left(\frac{xy^2}{2}\right)_0^x=\int_0^1\frac{x^3}{2}dx=\left(\frac{x^4}{2\times4}\right)_0^1=\frac{1}{8}.$

Correct option is (c)

Question 15. $\vec{F}=\frac{1}{x}\hat{i}+2y\hat{j}$, then $\vec{F}$ is a gradient of the following function **[H.C.U. 2014]**

(a) $f=-\frac{1}{x^2}+y^2+c$ (b) $f=y^2+c$ (c) $f=\log x+y^2+c$ (d) $f=2xy+c$

Soln: From option (c), $f=\log x+y^2+c$

$$\therefore\ \vec{\nabla}f=\frac{1}{x}\hat{i}+2y\hat{j}$$

Correct option is (c)

Question 16. The unit vector parallel to the resultant of the vectors $\vec{r}_1 = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{r}_2 = \hat{i} + 2\hat{j} + 3\hat{k}$ is given by **[H.C.U. 2015]**

(a) $\frac{1}{3}\hat{i} + \frac{1}{6}\hat{j} - \frac{1}{2}\hat{k}$ (b) $\frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$ (c) $\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$ (d) $3\hat{i} + 6\hat{j} - 2\hat{k}$

Soln: Resultant of $\vec{r}_1$ and $\vec{r}_2$ is $\vec{r} = \vec{r}_1 + \vec{r}_2 = 3\hat{i} + 6\hat{j} - 2\hat{k}$

Therefore, unit vector $\frac{\vec{r}}{|r|} = \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{3^2 + 6^2 + 2^2}} = \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$

Correct option is (b)

Question 17. If $T = \sin x \sin y \sin z$, $\nabla^2 T$ is given by

(a) 0 (b) $-2\sin x \sin y \sin z$ **[H.C.U. 2015]**

(c) $3\sin x \sin y \sin z$ (d) $-3\sin x \sin y \sin z$

Soln: $T = \sin x \sin y \sin z$

$$\nabla^2 T = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\sin x \sin y \sin z)$$

$$= -\sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z$$

$$= -3 \sin x \sin y \sin z$$

Correct option is (d)

Question 18. A force $\vec{F} = -A\left(x^2 i + yj\right)$ is rotational. The potential energy function associated with it is given by **[H.C.U. 2015]**

(a) $U = A\left(\frac{x^3}{3} + \frac{y^2}{2}\right)$ (b) $U = A\left(\frac{x^3}{3} + \frac{y^2}{2}\right)$ (c) $U = A\left(\frac{x^3}{3} + \frac{y^2}{2}\right)$ (d) $U = A\left(\frac{x}{3} + \frac{y^3}{2}\right)$

Soln: Since, $\vec{F} = -A(x^2\hat{i} + y\hat{j})$ is irrotational i.e. $\vec{\nabla} \times \vec{F} = 0$, therefore it can be always written as gradient of a scalar function. i.e. $\vec{F} = \vec{\nabla} u \Rightarrow -A\left(x^2\hat{i} + y\hat{j}\right) = -\frac{\partial u}{\partial x}\hat{i} - \frac{\partial u}{\partial y}\hat{j}$

$$\frac{\partial u}{\partial x} = +Ax^2 \Rightarrow u = +\frac{Ax^3}{3} \text{ and } \frac{\partial u}{\partial y} = +Ay \Rightarrow u = +\frac{Ay^2}{2}$$

Therefore, $u = A\left(\frac{x^3}{3} + \frac{y^2}{2}\right)$

Correct option is (a)

Question 19. Two vectors in Cartesian coordinates are defined as

$\vec{a} = \sin(t)\hat{i} + \hat{j} + \cos(t)\hat{k}$ and $\vec{b} = \sin(t)\hat{i} - \hat{j} + \cos(t)\hat{k}$

for some paramater t. Then these two vectors are said to be **[H.C.U. 2016]**

(a) parallel to each other (b) perpendicular to each other

(c) make an angle t between them (d) not related to each other

Soln: Here, $\vec{a}\cdot\vec{b} = (\sin t\,\hat{i} + \hat{j} + \cos t\,\hat{k}).(\sin t\,\hat{i} - \hat{j} + \cos t\,\hat{k}) = \sin^2 t - 1 + \cos^2 t = 0$

$\therefore$ $\vec{a}$ and $\vec{b}$ are perpendicular to each.

Correct option is (b)

Question 20. The value of the integral $\int_0^1 dx \int_0^x y\,dy$ is **[H.C.U. 2016]**

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{6}$ (d) $\frac{1}{8}$

Soln: $\int_0^1 dx \int_0^x dy = \int_0^1 dx \left(\frac{y^2}{2}\right)_0^x = \int_0^1 \frac{x^2}{2} dx = \left(\frac{x^3}{2\times 3}\right)_0^1 = \frac{1}{6}.$

Correct option is (c)

Question 21. Two vectors on a plane are given as $\vec{a} = 3\hat{i} + 5\hat{j}$ and $\vec{b} = 4\hat{i} + 10\hat{j}$. The plane is then defined by the vector **[H.C.U. 2016]**

(a) $50\,\hat{k}$ (b) $10\,\hat{k}$ (c) $0\,\hat{k}$ (d) $15\,\hat{k}$

Soln: Plane is $\vec{a}\times\vec{b} = (3\hat{i} + 5\hat{j}) \times (4\hat{i} + 10\hat{j}) = 10\,\hat{k}$

Correct option is (b)

Question 21. If $\vec{V} = \vec{\nabla} \times \vec{A}$ and given that $\vec{V} = -\hat{k}$, then the vector $\vec{A}$ will be given by **[H.C.U. 2017]**

(a) $-\hat{i} - y\hat{j} + \hat{k}$ (b) $\hat{i} + \hat{j} - \hat{k}$ (c) $5\hat{i} + \hat{j} - 2\hat{k}$ (d) $-y\,\hat{i} + \hat{j} + 5\hat{k}$

Soln: From option (d), $\vec{A} = y\,\hat{i} + \hat{j} + 5\,\hat{k} \Rightarrow \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & 1 & 5 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(0-1) = -\hat{k}$

Correct option is (d)

Question 22. The value of the surface integral $\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S}$, where $\vec{A} = \hat{i}y + \hat{j}z + \hat{k}$ and S is the surface defined by the paraboloid $z = 1 - x^2 - y^2$, with $z \geq 0$ is given by **[H.C.U. 2017]**

(a) $-\pi$ (b) π (c) 1 (d) -1

Soln:

Question 23. In plane polar coordinates, which of the following relation are true ? **[H.C.U. 2017]**

(a) $\frac{d\hat{r}}{d\theta} = \hat{\theta}, \frac{d\hat{\theta}}{d\theta} = -\hat{r}$ (b) $\frac{d\hat{r}}{d\theta} = -\hat{\theta}, \frac{d\hat{\theta}}{d\theta} = \hat{r}$ (c) $\frac{d\hat{r}}{d\theta} = \hat{r}, \frac{d\hat{\theta}}{d\theta} = -\hat{\theta}$ (d) $\frac{d\hat{r}}{d\theta} = -\hat{r}, \frac{d\hat{\theta}}{d\theta} = \hat{\theta}$

Soln: In plane polar coodinate system, we have

$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \Rightarrow \frac{\partial \hat{r}}{\partial \theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{\theta}$ and

$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \Rightarrow \frac{\partial \hat{\theta}}{\partial \theta} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} = -\hat{r}$

Correct option is (a)

BHU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. In order to determine uniquely a vector field, it is necessary to specify **[B.H.U. 2010]**

(a) its divergence only (b) its curl only (c) curl of its curl (d) its divergence and curl both

Soln: Correct option is (d)

Question 23. A vector field $\vec{A}$ is said to be non-conservative if **[B.H.U. 2010]**

(a) $\oint \vec{A} \cdot \vec{dl} = 0$ (b) $\vec{\nabla} \times \vec{A} = 0$ (c) $\vec{A} \times \vec{\nabla} f$ (d) $\vec{\nabla} \cdot \vec{A} = 0$

Soln: For non-conservative force $\vec{\nabla} \times \vec{A} \neq 0$ and $\oint \vec{A} \cdot d\vec{l} \neq 0$.

Therefore, options (a), (b) and (c) are incorrect.

Question 2. If two non-null vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$, then the angle between the two vectors is

(a) 0° (b) $\frac{\pi}{2}$ (c) π (d) $\frac{\pi}{3}$ **[B.H.U. 2010]**

Soln: $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$

$|\vec{A} + \vec{B}|^2 = |\vec{A} - \vec{B}|^2$

$A^2 + B^2 + 2AB\cos\theta = A^2 + B^2 - 2AB\cos\theta$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

Correct option is (b)

Question 3. A vector field $\vec{A}$ is said to be non-conservative if **[B.H.U. 2011]**

(a) $\oint \vec{A}\cdot d\vec{\ell}=0$ (b) $\vec{\nabla}\times\vec{A}=0$ (c) $\vec{A}=\vec{\nabla}f$ (d) $\vec{\nabla}\cdot\vec{A}=0$

Question 4. For an incompressible fluid $\vec{\nabla}(\rho\vec{v})$, where ρ is the density and $\vec{v}$ is the velocity, is

(a) always positive (b) always negative (c) always zero **[B.H.U. 2011]**
(d) sometimes positive and sometimes negative

Soln: For an in compressible fluid $\nabla(\rho\vec{V})$ is always zero.

Correct option is (c)

Question 5. The vector $\vec{r}$ is directed from the point $p'(x', y', z')$ to the point $P(x, y, z)$. The point P' is fixed and P is allowed to move. If $\hat{r}$ is a unit vector in the direction of $\vec{r}$, then **[B.H.U-2011]**

(a) $\vec{\nabla}\left(\frac{1}{r}\right)=-\frac{\hat{r}}{r^2}$ (b) $\vec{\nabla}\left(\frac{1}{r}\right)=\frac{\hat{r}}{r^2}$ (c) $\vec{\nabla}'\left(\frac{1}{r}\right)=-\frac{\hat{r}}{r^2}$ (d) $\vec{\nabla}'\left(\frac{1}{r}\right)=\frac{\hat{r}}{r^2}$

Soln: $\vec{r}=(x-x')\hat{i}+(y-y')\hat{j}+(z-z')\hat{k}$

$\vec{\nabla}\left(\frac{1}{r}\right)=\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{r}$ (Using spherical polar co-ordinate)

$=-\frac{\hat{r}}{r^2}$.

Correct option is (a)

Question 6. If for a scalar function ϕ, $\vec{\nabla}\phi=\frac{\vec{r}}{r^2}$, then ϕ is equal to **[B.H.U. 2012]**

(a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$ (c) $\ln(|r|)$ (d) $\ln(r)$

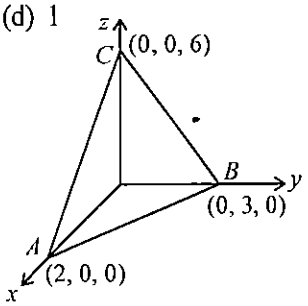
Soln: Since, $\vec{\nabla}\phi=\frac{\partial\phi}{\partial r}\hat{r}=\frac{\vec{r}}{r^2}=\frac{\hat{r}}{r}$. Therefore, $\frac{\partial\phi}{\partial r}=\frac{1}{r}$ integrating both side.

$\boxed{\phi=\ln(r)}$

Correct option is (d)

Question 7. Line integral of the vector $\vec{A}=(x+y)\hat{i}+(2x-z)\hat{j}+(y+z)\hat{k}$ along the sides of the triangle cut from the plane $3x+2y+z=6$ by the coordinate axes is **[B.H.U-2013]**

(a) 21 (b) 36 (c) −16 (d) 1



Soln: Given plane is $3x+2y+z=6 \Rightarrow \frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$

Given vector is $\vec{A}=(x+y)\hat{i}+(2x-z)\hat{j}+(y+z)\hat{k}$

Therefore, $\int\vec{A}\cdot d\vec{l}=\int_{AB}\vec{A}\cdot d\vec{l}+\int_{BC}\vec{A}\cdot d\vec{l}+\int_{CA}\vec{A}\cdot d\vec{l}$

Along AB, equation of AB, $y-0=\frac{3-0}{0-2}(x-2)\Rightarrow -2y=3x-6\Rightarrow dy=-\frac{3}{2}dx$ and $z=0$, $dz=0$.

$$\int_{AB} \vec{A}\cdot d\vec{l} = \int_{AB} (x+y)\,dx + (2x-z)\,dy$$

$$= \int_{x=2}^{0} \left(x - \frac{3}{2}x + 3\right)dx + (2x-0)\times -\frac{3}{2}dx$$

$$= \left(-\frac{1}{2}\,\frac{x^2}{2} + 3x - 3\frac{x^2}{2}\right)_2^0 = 1.$$

Along BC, equation of BC, $x = dx = 0$

$$z - 0 = \frac{6-0}{0-3}(y-3) \Rightarrow z = -2y+6 \Rightarrow dz = -2dy.$$

Therefore, $\int_{BC} \vec{A}\cdot d\vec{l} = \int_{BC} (2x-z)\,dy + (y+z)\,dz$

$$= \int_{y=3}^{0} -(-2y+6)\,dy + (y-2y+6)\times -2dy$$

$$= \left(\frac{2y^2}{2} - 6y + \frac{2y^2}{2} - 2y\right)_3^0 = 36.$$

Along CA, equation of CA, $y = dy = 0$

$$x - 0 = \frac{2-0}{0-6}(z-6) \Rightarrow x = -\frac{1}{3}z + 2 \Rightarrow dx = -\frac{1}{3}dz$$

$$\int_{CA} \vec{A}\cdot d\vec{l} = \int_{CA} (x+y)\,dx + (y+z)\,dz$$

$$= \int (x\,dx + z\,dz) = \int_{x=0}^{2} x\,dx + \int_{6}^{0} z\,dz$$

$$= \left(\frac{x^2}{2}\right)^2 + \left(\frac{z^2}{2}\right)_6^0 = 2 - 18 = -16.$$

Therefore, $\int \vec{A}\cdot d\vec{l} = 1 + 36 - 16 = 21.$

Correct option is (a)

Question 8. The following vector is orthogonal to the vector $a\hat{i} + b\hat{j}$ (with $a \neq b$)

(a) $a\hat{i} - b\hat{j}$ (b) $-a\hat{i} - b\hat{j}$ (c) $-b\hat{i} + a\hat{j}$ (d) $-b\hat{i} - a\hat{j}$ **[B.H.U-2016]**

Soln: Check option (c); $\left(a\hat{i} + b\hat{j}\right)\cdot\left(-b\hat{i} + a\hat{j}\right) = -ab + ab = 0$

Correct option is (c)

Question 9. If $\vec{r}$ is a position vector, the value of $\vec{\nabla}^2(\vec{r}\cdot\vec{r}) = \vec{\nabla}^2 r^2$ is equal to

(a) 6 (b) 0 (c) 3 (d) –3 **[B.H.U-2016]**

Soln: $\nabla^2 r^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}r^2\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\times 2r\right) = \frac{2}{r^2}\times 3r^2 = 6.$

Correct option is (a)

Question 10. $r^n\vec{r}$ is solenoidal for **[B.H.U 2017]**

(a) $n=3$ (b) $n=-3$ (c) $n=2$ (d) $n=-2$

Soln: If $r^n\vec{r}$ or $r^{n+1}\hat{r}$ is solenoidal, then $\vec{\nabla}\cdot\left(r^{n+1}\hat{r}\right)=0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\times r^{n+1}\right)=0$$

$$\frac{1}{r^2}(n+3)r^{n+2}=0$$

Therefore, $n+3=0 \Rightarrow n=-3$.

Correct option is (b)

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Chapter 2

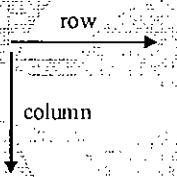
MATRICES

2.1 Basic Review of Matrices

Definition:
A square or rectangular array of numbers or functions is known as matrices. Individual numbers or functions are known as elements of the matrix.

Example: $\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}, \begin{bmatrix} 1-i & 2+i \\ 4+2i & x+iy \\ 2+4i & 3-2i \end{bmatrix}, \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$

Order of a matrix:
A matrix of order $m \times n$ has 'm' number of rows and 'n' number of columns.



Every element of matrix is characterised by a row index and a column index:
a_{ij} : 'i' → row index , 'j' → column index, so, a_{ij} is the element of i^{th} row and j^{th} column

If a matrix A contains m rows and n columns, then it is written as $A = [a_{ij}]_{m \times n}$

Basic Review of matrix operation:

(i) Two matrices A and B are said to be equal if they are of the same order and each element of one is equal to corresponding element of the other matrix i.e. $a_{ij} = b_{ij}$ for all values of 'i' and 'j'.

(ii) Two matrices A and B are said to be conformable for addition/substraction if they have same order.

(iii) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\lambda A = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$

(iv) Two matrices A and B are said to be comformable for multiplication if number of columns in A is equal to number of rows in B. Say, A matrix is of order '$m \times n$' and B matrix is of order '$p \times q$', then AB is possible only if $n = p$.

Various Type of Matrices:

(1) Square matrix: Matrix having number of rows equal to the number of columns i.e. $m = n$

Example: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$

(2) Diagonal matrix: Matrix having elements other than the principal diagonal elements are zero

i.e $a_{ij} = 0$ for $i \neq j$.

Example: $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

(3) Scalar matrix: Diagonal matrix having all principal diagonal elements to be same

i.e. $a_{ij} = 0$ for $i \neq j$ and a_{ii} is same for all i.

Example: $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

(4) Identity/unit matrix: Diagonal matrix having all principal diagonal elements equal to one.

i.e. $a_{ij} = 0$ for $i \neq j$ and a_{ii} is equal to 1 for all i.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(5) Row Matrix: Matrix having only one row.

Example: $\begin{bmatrix} a & b & c \end{bmatrix}_{1\times3}$

(6) Column matrix: Matrix having only one column.

Example: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}_{3\times1}$

(7) Null matrix: Matrix having all elements equal to zero i.e. $a_{ij} = 0$ for i, j

Example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

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(8) Upper triangular matrix: Matrix having elements below the principal diagonal equal to zero

i.e. $a_{ij} = 0$ for all $i > j$

Example: $\begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$

(9) Lower triangular matrix: Matrix having elements above the principal diagonal equal to zero

i.e. $a_{ij} = 0$ for all $i < j$

Example: $\begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix}$

(10) Periodic matrix: A square matrix 'A' for which $A^{k+1} = A$, is called periodic matrix of period k.

Example: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^5 = A$ i.e A is periodic matrix of period 4.

(11) Idempotent matrix: A square matrix 'A' for which $A^2 = A$, is called idempotent matrix. It is a periodic matrix of period 1. Any idempotent matrix will be either singular matrix or non-singular unit matrix.

Example: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

(12) Nilpotent matrix: A square matrix for which $A^p = 0$, is called nilpotent matrix of index 'p'. The trace and determinant of the nilpotent matrix is always zero.

Example: $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is a nilpotent matrix of index '2'.

(13) Involutory Matrix: A square matrix for which $A^2 = I$, is called involutory matrix. It is a self-inverse matrix.

Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Trace of a matrix

It is defined as the sum of the principal diagonal elements of the matrix i.e.

$$Tr(A) = a_{11} + a_{22} + a_{33} + \ldots\ldots\ldots\ldots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

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Properties:

(i) $Tr(A+B) = Tr(A) + Tr(B)$ (ii) $Tr(cA) = cTr(A)$ [where c is constant]

(iii) $Tr(AB) = Tr(BA)$ (iv) $Tr(ABCD) = Tr(BCDA) = Tr(CDAB) = Tr(DABC)$

Transpose of a matrix:

By interchanging the rows and columns of a matrix, we will get the transpose of the original matrix.

Example: $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$

Properties: (i) $(A^T)^T = A$, (ii) $(AB)^T = B^T A^T$

Transpose Conjugate of a Matrix:

It is obtained by taking the transpose of the matrix and then taking the complex conjugate of each element or vice versa i.e

$$A^* = \left(A^T\right)^* = \left(A^*\right)^T$$

Example: $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow A^{\dagger}=\begin{bmatrix} a^* & d^* & g^* \\ b^* & e^* & h^* \\ c^* & f^* & i^* \end{bmatrix}$

Properties: (i) $\left(A^{\dagger}\right)^{\dagger}=A$, (ii) $(AB)^{\dagger}=B^{\dagger}A^{\dagger}$

Example 1: If each element of '$k\times k$' square matrix A can be chosen as a Boolean Variable, then one can construct 'M' number of different combinations of matrix 'A'. Therefore, 'M' is equal to

(a) $2k$ (b) 2^{k^2} (c) k^2 (d) 2^k

Soln: Total number of elements $=k^2$

Each of the elements can be chosen to be 0 or 1 i.e. in 2 ways

Therefore, entire k^2 number of elements can be chosen in $2\times 2\times \ldots\ldots\ldots k^2$ times $=2^{k^2}$

Correct option is (b)

Example 2: Which of the following statements is NOT CORRECT?

(a) Trace of a unit matrix is always a fixed natural number
(b) Trace of the sum of two matrices is equal to sum of their traces
(c) Trace of the scalar matrix is some multiple of its order
(d) $Tr(AB)=Tr(BA)$

Soln: Trace of a unit matrix of order 'n' is equal to 'n' i.e. it is natural number but not fixed.

Correct option is (a)

Example 3: Consider the matrix $B=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then B^{1023} equals

(a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Soln: $B=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow B^2=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}=-I$ and $B^4=\left(B^2\right)^2=(-I)^2=I$

Therefore, $B^{1023}=B^{1020}.B^3=\left(B^4\right)^{255}.B^3=I.B^3=B^3=B^2.B=(-I).B=-B=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Correct option is (b)

Symmetric matrix: Matrix for which $A^T=A$ i.e. $a_{ij}=a_{ji}$.

Example: $\begin{bmatrix} a & b & c \\ b & c & d \\ c & d & e \end{bmatrix}$

Note: The number of independent components of a $n\times n$ symmetric matrix is $\frac{1}{2}n(n+1)$.

Skew-symmetric matrix: Matrix for which $A^T=-A$ i.e. $a_{ij}=-a_{ji}$.

Example: $\begin{bmatrix} 0 & b & c \\ -b & 0 & d \\ -c & -d & 0 \end{bmatrix}$

Note: (i) The number of independent components of a $n\times n$ antisymmetric matrix is $\frac{1}{2}n(n-1)$.

(ii) The principal diagonal elements of $n\times n$ skew-symmetric matrix are always zero.

- Every square matrix can be written as the sum of symmetric and antisymmetric matrix.

$$A=\frac{1}{2}(A+A^T)+\frac{1}{2}(A-A^T)$$

- If A and B are symmetric matrices, then AB will be symmetric if and only if A and B commute.

Hermitan matrix: Matrix for which $A^\dagger=A$ i.e. $a_{ij}=a_{ji}^*$.

Example: $A=\begin{bmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{bmatrix}$ where $a,b,c\in R$

Note: (i) Principal diagonal elements of Hermitian matrix are always real.

(ii) Minimum number of real elements for the most general $n\times n$ hermitian matrix is n

Skew-Hermitan matrix: Matrix for which $A^\dagger=-A$ i.e. $a_{ij}=-a_{ji}^*$.

Example: $A=\begin{bmatrix} a & d & e \\ -d^* & b & f \\ -e^* & -f^* & c \end{bmatrix}$ where a,b,c can be either zero or purely imaginary.

Note: (i) Principal diagonal elements of a skew-hermitian matrix are either zero or purely imaginary.

(ii) Minimum number of real elements in a most general $n\times n$ skew hermitian matrix is zero.

(iii) If A is Hermitian matrix, iA and $-iA$ will be anti-Hermitian.

(iv) If A is Hermitian matrix, kA (k is real scalar) will be also Hermitian.

Example 4: The product MN of two Hermitian matrices M and N is anti-Hermitian. It follows that

(a) $\{M,N\}=0$ (b) $[M,N]=0$ (c) $M^\dagger=N$ (d) $M^\dagger=N^{-1}$

Soln: If MN is anti-Hermitian, then

$$(MN)^\dagger=-(MN)\Rightarrow N^\dagger M^\dagger=-MN\Rightarrow NM=-MN\Rightarrow MN+NM=0\Rightarrow\{M,N\}=0$$

Correct option is (a)

Example 5: To convert a Hermitian matrix into Skew-Hermitian one, the Hermitian matrix must be multiplied by

(a) -1 (b) i (c) $-i$ (d) 1

Soln. To convert a Hermitian matrix into Skew-Hermitian we need to multiply it by an imaginary number.

Correct options are (b) and (c)

Example 6: If A is a $n \times n$ square matrix. Then, the correct statements is/are :

(a) If A is Skew-symmetric, then $Tr(A) = 0$

(b) If A is Hermitian, then $Tr(A)$ is a real number

(c) If A is Skew-symmetric, then $Tr(A)$ is zero

(d) If A is symmetric with distinct eigenvalues, then it is diagonalizable

Soln. All options are correct.

Correct options are (a), (b), (c) and (d)

Example 7: If $A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$ is an orthogonal matrix, then

(a) $a^2 + b^2 + c^2 = 1$ (b) $a + b + c = 1$ (c) $a^2 + b^2 + c^2 = 0$ (d) $a + b + c = 0$

Soln. Since, A is orthogonal, hence each row or column is a unit vector and every rows and columns are orthogonal to each.

Therefore, $\vec{R}_1 = a\hat{i} + b\hat{j} + c\hat{k}$ and $\vec{R}_2 = \hat{i} + a\hat{j} + c\hat{k}$.

Therefore, $\left|\vec{R}_1\right| = \sqrt{a^2 + b^2 + c^2} = 1 \Rightarrow a^2 + b^2 + c^2 = 1$ and $\vec{R}_1 \cdot \vec{R}_2 = 0 \Rightarrow ab + bc + ca = 0$.

Now, $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 1 + 0 \Rightarrow a + b + c = 1$.

Correct options are (a) and (b)

Example 8: For any operator A, $i\left(A^{\dagger} - A\right)$ is :

(a) hermitian (b) anti-hermitian (c) unitary (d) orthogonal

Soln. $\left[i\left(A^{\dagger} - A\right)\right]^{\dagger} = (i)^{\dagger}\left[\left(A^{\dagger}\right)^{\dagger} - (A)^{\dagger}\right] = i\left(A^{\dagger} - A\right)$

Correct option is (a)

Example 9: For arbitrary matrices E, F, G and H, if $EF - FE = 0$, then Trace $(EFGH)$ is equal

(a) Trace (HFEG) (b) Trace (E), Tace (F), Trace (G), Trace (H)

(c) Trace (GFEH) (d) Trace (EGHF)

Soln. $Tr(EFGH) = Tr(HEFG) = Tr(HFEG)$

Correct option is (a)

Example 10: An unitary matrix $\begin{bmatrix} ae^{i\alpha} & b \\ ce^{i\beta} & d \end{bmatrix}$ is given, where a, b, c, d, α and β are real. The inverse of the matrix is

(a) $\begin{bmatrix} ae^{i\alpha} & -ce^{i\beta} \\ b & d \end{bmatrix}$ (b) $\begin{bmatrix} ae^{i\alpha} & ce^{i\beta} \\ b & d \end{bmatrix}$ (c) $\begin{bmatrix} ae^{i\alpha} & b \\ ce^{i\beta} & d \end{bmatrix}$ (d) $\begin{bmatrix} ae^{-i\alpha} & ce^{-i\beta} \\ b & d \end{bmatrix}$

Soln. For unitary matrix, $U^{-1} = U^{+} = \begin{bmatrix} a\,e^{-i\alpha} & c\,e^{-i\beta} \\ b & d \end{bmatrix}$

Correct option is (d)

Singular matrix: Matrix having zero determinant.

Example: $\begin{bmatrix} 2 & 1 & 5 \\ 3 & 5 & 7 \\ 2 & 1 & 5 \end{bmatrix}$

Non-singular matrix: Matrix having non-zero determinant.

Example: $\begin{bmatrix} 3 & 1 & 1 \\ 6 & -2 & 4 \\ 0 & 3 & -7 \end{bmatrix}$

Adjoint of a Matrix: Adjoint of a matrix is defined as the transpose of co-factor matrix.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $Adj\ A = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$

Note: (i) $(Adj\ A).\ A = A.(Adj\ A) = |A|\ I_n$

(ii) If A is a square matrix of order 'n' having determinant 'm', then $|\ Adj\ A\ | = m^{n-1}$.

Inverse of a Matrix:

Let, A be a square matrix of order 'n', and B be a square matrix of order 'n' such that $AB = BA = I_n$,

then $B = A^{-1}$ = inverse of matrix $A = \frac{adj\ A}{|A|}$; Therefore, A is called invertible matrix.

Note: (i) Any invertible matrix possesses an unique inverse.

(ii) The necessary and sufficient condition for a square matrix A to posses an inverse is that it is non-singular.

(iii) $(ABC....)^{-1} = ... C^{-1}B^{-1}A^{-1}$

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Example 11: If 'I' is an identity matrix of order '$n \times n$' and adj $(2I) = 2^k$ I, then 'k' is equal to

(a) 1 (b) 2 (c) n-1 (d) n

Soln. $A.(adj\ A) = |A|I \Rightarrow (2I).(adj\ 2I) = |2I|I \Rightarrow 2(adj\ 2I) = 2^n I \Rightarrow (adj\ 2I) = 2^{n-1}I \Rightarrow k = n-1$

Correct option is (c)

Example 12: If 'A' is a 3×3 matrix with determinant 3, then the determinant of $adj.[adj.[adj.(A^{-1})]]$ is equal to

(a) $\frac{1}{81}$ (b) $\frac{1}{729}$ (c) $\frac{1}{6561}$ (d) $\frac{1}{243}$

Soln: $|A|=3 \Rightarrow |A^{-1}|=\frac{1}{3} \Rightarrow |adj\,(A^{-1})|=\left(\frac{1}{3}\right)^{3-1}=\frac{1}{9} \Rightarrow |adj\{adj\,(A^{-1})\}|=\left(\frac{1}{9}\right)^{3-1}=\frac{1}{81}$

$\Rightarrow |adj[adj\{adj\,(A^{-1})\}]|=\left(\frac{1}{81}\right)^{3-1}=\frac{1}{6561}$

Correct option is (c)

Example 13: Suppose A is $n \times n$ square matrix having determinant p. Which of the following statements is/ are true for $A(\text{adj}A)$

(a) It has determinant p^{n-1} (b) It is an unit matrix of order n

(c) It is a scalar matrix of order n (d) It has determinant p^n

Soln. Taking determinant of $A(\text{adj}A)=|A(\text{adj}\,A)|$

$=|A||\text{adj}\,A|$

$=p\,p^{n-1} \quad \left(\because |\text{adj}\,A|=|A|^{n}\right)$

$=p^n$

Also, $A(\text{adj}\,A)=(\text{adj}\,A)A=|A|\,I=pI$

Hence, it is a scalar matrix of order n.

Correct options are (c) and (d)

Example 14: If A and B are 3×3 square matrix such that $|A|=2$ and $|B|=3$, then the value of $|\text{adj}(AB)|$ is _____.

Soln. We have $|\text{adj}(AB)|=|\text{adj}\,A||\text{adj}\,B|=|A|^{3-1}|B|^{3-1}=2^2 \times 3^2=36.$

Correct answer is (36)

Example 15: Which one of the following is the inverse of the matrix $\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$?

(a) $\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$ (b) $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ (c) $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ (d) $\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$

Soln. $A^{-1}=\frac{adjA}{|A|}=\frac{1}{1}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

Correct option is (c)

Orthogonal Matrix: A square matrix 'A' is said to be orthogonal if $AA^T=A^TA=I$

Note: (i) Determinant of an orthogonal matrix will be either +1 or –1.

(ii) $A^T=A^{-1}$

(iii) Each row and column of an orthogonal matrix is a normalized vector.

(iv) Any two rows or any two columns of an orthogonal matrix are orthogonal to each other.

(v) If A and B are orthogonal matrices, then AB and BA will be always orthogonal.

Example: $A=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$

Unitary matrix: A square matrix 'A' is said to be orthogonal if $AA^{\dagger}=A^{\dagger}A=I$

Note: (i) Determinant of an orthogonal matrix will be of unit modulus..

(ii) $A^{\dagger}=A^{-1}$

(iii) Each row and column of an unitary matrix is a normalized vector.

(iv) Any two rows or any two columns of an unitary matrix are orthogonal to each other.

(v) If A and B are unitary matrices, then AB and BA will be always unitary.

Example: $A=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}$

Example 16: The matrix A, defined by $A=\begin{pmatrix}1 & 0 & 0\\ 0 & a & b\\ 0 & -b & a\end{pmatrix}$ is orthogonal if

(a) $a=1, b=-1$ (b) $a=1/\sqrt{2}, b=-1/\sqrt{2}$ (c) $a=1/\sqrt{2}, b=-i/\sqrt{2}$ (d) $a=1,\ b=1$

Soln: Applying the condition for the orthogonal matrices i.e. $AA^{T}=I$ we get, $a^2+b^2=1$

Only option (b) satisfies this.

Correct option is (b)

Example 17: Given a 2 x 2 unitary matrix satisfying $UU^{\dagger}=U^{\dagger}U=I$ with $\det U=e^{i\varphi}$, one can construct a unitary matrix V $\left(V^{\dagger}V=VV^{\dagger}=I\right)$ with det V = 1 from it by

(a) Multiplying U by $e^{-i\varphi/2}$ (b) Multiplying a single element of U by $e^{-i\varphi}$

(c) Multiplying any row or column by $e^{-i\varphi/2}$ (d) Multiplying U by $e^{-i\varphi}$

Soln. Let $U=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $\det U=ad-bc=e^{i\phi}$ (Given)

Let, $V=e^{-i\phi/2}U \quad\Rightarrow\quad V=\begin{pmatrix}e^{-i\phi/2}a & e^{-i\phi/2}b\\ e^{-i\phi/2}c & e^{-i\phi/2}d\end{pmatrix}$

$\det V=e^{-i\phi}ad-e^{-i\phi}bc=e^{-i\phi}\left(ad-bc\right)=e^{-i\phi}e^{i\phi}=1$

We have to multiply U with $e^{-i\phi/2}$ to get V with determinant = 1

Correct answer is (a)

Example 18: The matrix $A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}$ is

(a) orthogonal (b) symmetric (c) anti-symmetric (d) unitary

Soln. $AA^{+}=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$

So, A is an unitary matrix

Correct option is (d)

2.2 Eigenvalues and Eigenvectors:

Consider the following linear transformation, $AX=\lambda X \Rightarrow (A-\lambda I)X=0$

The values of λ for which the above equation has non zero solution of X $(X\neq 0)$ are called eigen values or characteristic roots of matrix A and corresponding non-zero solution X is called eigenvector or characteristic vector of matrix A corresponding to that value of λ.

Here, $(A-\lambda I)$ is called Characteristic matrix of matrix A, $|A-\lambda I|$ is called Characteristic polynomial of matrix A, and $|A-\lambda I|=0$ is called Characteristic equation or eigen equation of matrix A.

Example 19: Find the characteristic roots or eigenvalues and eigenvectors of matrix.

$$A=\begin{pmatrix}1 & 2 & 3\\ 0 & 2 & 3\\ 0 & 0 & 2\end{pmatrix}$$

Soln: Eigenvalue equation:

$$|A-\lambda I|=0 \Rightarrow \begin{vmatrix}1-\lambda & 2 & 3\\ 0 & 2-\lambda & 3\\ 0 & 0 & 2-\lambda\end{vmatrix}=0 \Rightarrow (1-\lambda)\left[(2-\lambda)^2-0\right]=0 \Rightarrow \lambda=1,2,2$$

Calculation of eigenvectors:

(i) $\lambda_1=1$: $AX_1=\lambda_1 X_1$

$$\Rightarrow \begin{pmatrix}1 & 2 & 3\\ 0 & 2 & 3\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=1\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} \Rightarrow \begin{pmatrix}x_1+2x_2+3x_3\\ 2x_2+3x_3\\ 2x_3\end{pmatrix}=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} \Rightarrow \begin{aligned}x_1+2x_2+3x_3&=x_1\\ 2x_2+3x_3&=x_2\\ 2x_3&=x_3\end{aligned}$$

Solving these equations, we get

$x_3=0,\ x_2=0$ & let $x_1=k$. Therefore, $X_1=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}k\\ 0\\ 0\end{bmatrix}$

If this eigen vector is normalized to unity then,

$$X_1^{\dagger}X_1=1 \Rightarrow \begin{bmatrix}k^{*} & 0 & 0\end{bmatrix}\begin{bmatrix}k\\ 0\\ 0\end{bmatrix}=1 \Rightarrow |k|^2=1 \Rightarrow k=1 \Rightarrow X_1=\begin{bmatrix}k\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

(ii) $\lambda_2=\lambda_3=2$:

Let the eigen vector is $X_2=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ $\quad\therefore\quad AX_2=\lambda_2X_2$

$$\Rightarrow\quad\begin{pmatrix}1&2&3\\0&2&3\\0&0&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=2\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\quad\Rightarrow\quad\begin{pmatrix}x_1+2x_2+3x_3\\2x_2+3x_3\\2x_3\end{pmatrix}=\begin{pmatrix}2x_1\\2x_2\\2x_3\end{pmatrix}\Rightarrow\begin{matrix}x_1+2x_2+3x_3=2x_1\\2x_2+3x_3=2x_2\\2x_3=2x_3\end{matrix}$$

it gives $x_1=2x_2$ & x_3 can be arbitary, we choose x_3 to be zero.

Let $x_2=k\ \Rightarrow x_1=2k\ \therefore\ X_2=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}2k\\k\\0\end{pmatrix}$

If the eigen vector is normalized to unity, then

$X_2^+X_2=1\Rightarrow\begin{pmatrix}2k^* & k^* & 0\end{pmatrix}\begin{pmatrix}2k\\k\\0\end{pmatrix}=1\Rightarrow5|k|^2=1\Rightarrow k=\frac{1}{\sqrt{5}}$, therefore, $X_2=\begin{pmatrix}2/\sqrt{5}\\1/\sqrt{5}\\0\end{pmatrix}$

Example 20: If $\begin{bmatrix}-4.5\\-4\\1\end{bmatrix}$ is an eigenvector of $\begin{bmatrix}8&-4&2\\4&0&2\\0&-2&-4\end{bmatrix}$, the eigenvalue corresponding to the eigenvector is

(a) 1 (b) 4 (c) –4.5 (d) 6

Soln. If λ is an eigenvalue, then $AX=\lambda X$

$$\Rightarrow\begin{bmatrix}8&-4&2\\4&2&2\\0&-2&-4\end{bmatrix}\begin{bmatrix}-4.5\\-4\\1\end{bmatrix}=\lambda\begin{bmatrix}-4.5\\-4\\1\end{bmatrix}$$

$$\begin{bmatrix}-18\\-16\\4\end{bmatrix}=\lambda\begin{bmatrix}-4.5\\-4\\1\end{bmatrix}$$

$\Rightarrow\ \lambda=4.$

Correct option is (b)

Example 21: Consider the matrix $P=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, then the correct statements is/are ?

(a) eigenvalues of P are i and $-i$ (b) eigenvalues of P are $1-i$ and $-1+i$

(c) eigenvectors of P are $\begin{pmatrix}1\\i\end{pmatrix}$ and $\begin{pmatrix}-1\\i\end{pmatrix}$ (d) eigenvectors of P are $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$

Soln. Characteristic equation, $|P-\lambda I|=0$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

Eigenvector for $\lambda = i$,

$$(A - \lambda I)X = 0 \Rightarrow \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$-iX + y = 0 \Rightarrow X = -iy$

If, $y = -1$, $x = i$

Therefore, $X_{\lambda=i} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Eigenvector for $\lambda = -i$, $\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$ix + y = 0 \Rightarrow x = -y$

Hence if $y = 1$, $x = i$. Therefore, $X_{\lambda=-i} = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

Correct options are (a) and (c)

Example 22: The eiegenvalues of the anti-symmetric matrix $A = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$ where n_1, n_2, n_3 are components of an unit vector, are

(a) 0, -i, i (b) 0, 1, -1 (c) 0, 1+i, -1-i (d) 0, 0, 0

Soln. Characteristic equation of the given matrix : $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & -n_3 & n_2 \\ n_3 & -\lambda & -n_1 \\ -n_2 & n_1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 + \lambda\left(n_1^2 + n_2^2 + n_3^2\right) = 0$$

Since, n_1, n_2, n_3 are the components of an unit vector, so $n_1^2 + n_2^2 + n_3^2 = 0$

$\Rightarrow \lambda^3 + \lambda = 0 \Rightarrow \lambda = 0, i, -i$

Correct answer is (a)

Example 23: The eigenvalues of the matrix $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ are

(a) +1 and +1 (b) zero and +1 (c) zero and +2 (d) −1 and +1

Soln. Eigenvalue equation : $\begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 1)^2 + i^2 = 0 \Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda = 0, 2$

Correct option is (c)

Example 24: The eigenvalues of the matrix $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ are

(a) +1 and +1 (b) zero and +1 (c) zero and +2 (d) –1 and +1

Soln. Eigenvalue equation: $\begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-1)^2 + i^2 = 0$

$\Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda = 0, 2$

Correct option is (c)

Example 25 : The eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ are

(a) 6, 1 and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (b) 2, 5 and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) 6, 1 and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (d) 2, 5 and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Soln. $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$; eigenvalue equation: $|A - \lambda I| = 0$

$\Rightarrow \quad \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-2)(\lambda-5) - 4 = 0 \Rightarrow \lambda^2 - 7\lambda + 6 = 0 \Rightarrow \lambda = 6, 1$

Now, $(A - \lambda I)X = 0$

$\Rightarrow \quad \begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

Putting $\lambda = 6 \Rightarrow \begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow -x_1 + 4x_2 = 0 \Rightarrow x_1 = 4x_2$

So, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Putting $\lambda = 1 \Rightarrow \begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow 4x_1 + 4x_2 = 0 \Rightarrow x_1 = -x_2$

So, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Correct option is (a)

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Example 26: The eigenvalues of the matrix $A=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$ are

(a) real and distinct (b) complex and distinct
(c) complex and coinciding (d) real and coinciding

Soln. Eigenvalue equation : $\begin{vmatrix}-\lambda & i\\ i & -\lambda\end{vmatrix}=0 \Rightarrow \lambda^2-i^2=0 \Rightarrow \lambda=\pm i$

Correct option is (b)

Note:(i) Sum of the eigenvalues of a matrix is equal to trace of the Matrix i.e.

$\lambda_1+\lambda_2+\lambda_3+.....+\lambda_n=Tr(A)$

(ii) Product of the eigenvalues of a matrix is equal to determinant of the matrix i.e.

$\lambda_1.\lambda_2.\lambda_3.....\lambda_n=Det(A)$

(iii) Any square matrix A and its transpose have same eigen values.

(iv) If $\lambda_1, \lambda_2, \lambda_3, ...$ are the eigenvalues of A, then

(a) eigenvalues of the matrix kA will be $k\lambda_1, k\lambda_2, k\lambda_3.....$

(b) eigenvalues of the matrix A^{-1} will be $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}.....$

(c) eigenvalues of the matrix A^m will be $\lambda_1^m, \lambda_2^m, \lambda_3^m ....$

(v) Eigenvalues of a real symmetric matrix or a Hermitian matrix are always real.

(vi) Eigen values of a real skew-symmetric matrix or a skew-Hermitian matrix are either zero or purely imaginary.

(vii) Eigen values of an real orthogonal matrix or Unitary matrix are of unit modulus.

(viii) Eigen values of a diagonal matrix or a upper triangular matrix or a lower triangular matrix are the principal diagonal elements.

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(ix) Any two eigen vectors corresponding to two distinct eigen values of Hermitian matrix and a Unitary matrix are orthogonal to each other.

(x) Eigenvalues of a matrix is invariant under similarity transformation.

(xi) Eigenvalues of a nilpotent matrix are alwayszero.

(xii) Eigenvalues of idempotent matrix are either zero or unity.

(xiii) Consider a $n\times n$ matrix having all elements equal to one. One of the eigenvalue of the matrix will be equal to order of the matrix and all other eigenvalues are zero i.e $n,0,0,0,.....$

(xiv) Consider a $n\times n$ matrix having rows and columns which are scalar multiple of a particular row and column respectively. One of the eigenvalue of the matrix will be equal to trace of the matrix and all other eigenvalues are zero i.e trace,$0,0,0,.....$

Example 27: If $A=\begin{bmatrix}1 & 2 & -3\\ 0 & 3 & 2\\ 0 & 0 & -2\end{bmatrix}$, find the eigenvalues of $3A^3+5A^2-6A+2I$.

Soln: A is an upper triangular matrix, therefore eigenvalues of matrix A will be $\lambda=1, 3, -2$

Eigen values of $3A^3 = 3, 81, -24$

Eigen values of $5A^2 = 5, 45, 20$

Eigen values of $6A = 6, 18, -12$

Eigen values of $2I = 2, 2, 2$

Eigenvalue of given expression will be

$$(3+5-6+2)=4,(81+45-18+2)=110,(-24+20+12+2)=10$$

Example 28: The eigenvalues of $\begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$ are

(a) 1, 3 (b) –1, –6 (c) 1, –6 (d) –2, –5

Soln: Sum of the eigenvalues is equal to the trace of matrix i.e. $\lambda_1 + \lambda_2 = -7$

Product of the eigenvalues is equal to the determinant of matrix i.e. $\lambda_1\lambda_2 = 6$

Only option (b) satisfies this.

Therefore, eigenvalues are = –1, –6

Correct option is (b)

Example 29: If trace of matrix M of order 3, is 13 and its determinant is 36, then the largest eigenvalue of M is equal to

(a) 4 (b) 6 (c) 9 (d) 12

Soln: Sum of the eigenvalues = trace of the matrix i.e. $\lambda_1 + \lambda_2 + \lambda_3 = 13$

Product of the eigenvalues = determinant of the matrix i.e. $\lambda_1\lambda_2\lambda_3 = 36$

Only the combiantion $\lambda_1 = 9, \lambda_2 = 2, \lambda_3 = 2$ satisfies the both conditions simultaneously.

Correct option is (c)

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Example 30: What are the number of zero eigenvalues of matrix $\begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_5^2 \end{pmatrix}$?

Soln: For the given matrix, Second row $= \frac{a_2}{a_1} \times$ first row & third row $= \frac{a_3}{a_1} \times$ first row

Therefore, the eigenvalues of the matrix = trace of the matrix, $0, 0 = \left(a_1^2 + a_2^2 + a_3^2\right), 0, 0$

Example 31: The eigenvalues of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ are given by λ_1, λ_2 *and* λ_3. Which one of th following statement is **NOT TRUE**?

(a) $\lambda_1 + \lambda_2 + \lambda_3 = 0$ (b) $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 0$

(c) All eigenvalues are real (d) Product of the eigenvalues is 1

Soln: We have a matrix, $A=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}$

Eigenvalue equation is,

$$\therefore \quad |A-I\lambda|=0 \Rightarrow \begin{vmatrix}-\lambda & 1 & 0\\ 0 & -\lambda & 1\\ 1 & 0 & -\lambda\end{vmatrix}=0$$

$$\Rightarrow \quad -\lambda\left(\lambda^2-0\right)+1(1) \Rightarrow \lambda^3=1$$

$$\Rightarrow \quad \lambda_1=1, \lambda_2=\omega, \lambda_3=\omega^2$$

Therefore, all eigen values are not real.

Correct option is (c)

Example 32 : The matrix given by:

$$\begin{pmatrix}3 & 2i & 1\\ -2i & 4 & 1+2i\\ 1 & 1-2i & 5\end{pmatrix}$$

(a) Has one real and one complex eigenvalues (b) Has two real and one complex eigenvalues
(c) Has three real eigenvalues (d) Has three complex eigenvalues

Soln: Since all $a_{ij}=a_{ji}^{*}$ of the given matrix. So, the matrix is Hamilton matrix.

So, all the eigen values are real.

Correct option is (c)

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Example 33 : The eigenvalues of $\begin{bmatrix}5 & 6 & 17\\ 0 & -19 & 23\\ 2 & 2 & 37\end{bmatrix}$ are

(a) –19, 5, 37 (b) 19, –5, –37 (c) 2, –3, 7 (d) 3, –5, 37

Soln. Since, given matrix is an upper triangular matrix whose eigenvalues are principle diagonal element.
Hence, eigenvalues are 5, –19, 37.
Correct option is (a)

Example 34 : If one of the eigenvalues of $A_{n\times n}$ is zero, if implies

(a) The solution to $[A][X]=[C]$ system of equation is zero.

(b) The determinant of $[A]$ is zero.

(c) The solution to $[A][X]=[0]$ system of equation is trivial.

(d) The determinant of $[A]$ is non-zero.

Soln. As we know, determinant of a matrix = product of eigenvalues.
Hence, if any of one eigenvalue is zero, determinant of matrix = 0.
Correct option is (b)

Example 35 : Given that matrix $A=\begin{bmatrix}8 & -4 & 2\\ 4 & 0 & 2\\ 0 & -2 & -3\end{bmatrix}$ has an eigenvalue 4 with the corresponding eigenvectors of $X=\begin{bmatrix}-4.5\\ -4\\ 1\end{bmatrix}$, then $A^5[X]$ is

(a) $\begin{bmatrix}-18\\ -16\\ 4\end{bmatrix}$ (b) $\begin{bmatrix}-4.5\\ -4\\ 1\end{bmatrix}$ (c) $\begin{bmatrix}-4608\\ -4096\\ 1024\end{bmatrix}$ (d) $\begin{bmatrix}-0.0043\\ -0.003\\ 0.0097\end{bmatrix}$

Soln. As we know, if λ is an eigenvalue of matrix A and X is corresponding eigenvector, then $AX=\lambda X$ or $A^5X=\lambda^5X$.

Hence, $A^5X=\lambda^5X=4^5\begin{bmatrix}-4.5\\ -4\\ 1\end{bmatrix}=\begin{bmatrix}-4608\\ -4096\\ 1024\end{bmatrix}$.

Correct option is (c)

Example 36 : If matrix T has the eigenvalue λ with an associated eigenvector $\vec{V}$, then $\vec{V}$ is an eigenvector of $cT+dI$ associated with eigenvalue (here c and d are constants).

(a) $d\lambda-c$ (b) $c\lambda+d$ (c) $c\lambda-d$ (d) $d\lambda+c$

Soln. Since, $\vec{V}$ is an eigenvector of $(cT+dI)$.

Hence, $(cT+dI)\vec{V}=cT\vec{V}+dI\vec{V}$

$=c\lambda\vec{V}+d\vec{V}\quad(\because T\vec{V}=\lambda\vec{V})$

$=(c\lambda+d)\vec{V}$

Hence, eigenvalue of $(cT+dI)$ is $(c\lambda+d)$.

Correct option is (b)

Example 37 : Let A be a 3×3 real orthogonal matrix with $\det(A)=1$, if one of the eigenvalues is $\frac{-1+\sqrt{3}i}{2}$, then choose the correct statement(s) :

(a) Matrix A will have all eigenvalues real.
(b) Matrix A will have a complex conjugate eigenvalue.
(c) One of the eigenvalue of A is 1
(d) Product of the eigenvalues will be 1

Soln. Since, A is a real matrix and $\frac{-1+\sqrt{3}i}{2}$ is a complex eigenvalue, it's conjugate $\frac{-1-\sqrt{3}i}{2}$ is also an eigenvalues of A.

Also, product of eigenvalues = $\det(A)$

$\frac{-1+\sqrt{3}i}{2}\times\frac{-1-\sqrt{3}i}{2}\times\lambda=i\Rightarrow\lambda=1$

Correct options are (b), (c) and (d)

Example 38 : Consider a 3×3 matrix of the following form

$$\begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix}$$

Which of the following statements is/are TRUE ?

(a) has one degenerate eigenvalue (b) has no degenerate eigenvalue

(a) is singular in matrix (d) is trace less

Soln. $A = \begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix}$; Here, $R_2 \to \frac{a_2}{a_1}R_1$ and $R_3 \to \frac{a_3}{a_1}$

Hence, eigenvalue of A are $\left(a_1^2 + a_2^2 + a_3^2\right), 0, 0$

Therefore, '0' is degenerate eigenvalue.

Also, $\det(A) = 0$

Correct options are (a) and (c)

Example 39 : Consider a $n \times n$ square matrix A whose elements a_{ij} are given by

$$a_{ij} = \begin{cases} i^2 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

Now choose correct statement(s) about 'A'

(a) Trace of A is equal to 30 for $n = 4$ (b) Determinant of A is $(n!)^2$

(c) Eigenvectors of A are linearly independent (d) Eigenvectors of A are linearly dependent.

Soln. $A = \begin{bmatrix} 1^2 & 0 & & \cdots & 0 \\ 0 & 2^2 & & \cdots & 0 \\ 0 & 0 & 3^2 & \cdots & 0 \\ & & & & n^2 \end{bmatrix}$; It is a diagonal matrix

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Therefore, trace of $A = 1^2 + 2^2 + 3^2 + \cdots n^2 = \frac{n(n+1)(2n+1)}{6}$.

Put $n = 4$, trace of $A = 30$

Determinant of $A = 1^2 \cdot 2^2 \cdot 3^2 \cdot \ldots\ldots n^2 = (n!)^2$

Since, alll eigenvalues are distinct. Hence eigenvectors are linearly independent.

Correct options are (a), (b) and (c)

Example 40 : If $A = \begin{bmatrix} 1/7 & 3/7 & 3/7 \\ 3/7 & 1/7 & 3/7 \\ 3/7 & 3/7 & 1/7 \end{bmatrix}$, then the value of $\lim_{n \to \infty} (Tr(A^n))$ is equal to _____.

Soln. Matrix A is symmetric, hence diagonalizable.

Eigenvalues of matrix A are $1, -\frac{2}{7}, -\frac{2}{7}$

Since, the eigenvalue of A^n are $(1)^n, \left(-\frac{2}{7}\right)^n, \left(-\frac{2}{7}\right)^n$.

Therefore, $Tr(A^n) = 1^n + \left(-\frac{2}{7}\right)^n + \left(-\frac{2}{7}\right)^n$

$\Rightarrow \lim_{n\to\infty} Tr(A^n) = 1$

Correct answer is (1)

Example 41 : Consider the matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

A. The eigenvalues of M are

(a) 0,1,2 (b) 0,0,3 (c) 1,1,1 (d) -1,1,3

Soln. Eigenvalues of a n x n matrix having all elements equal to one, are n,0,0,0,..... respectively.

Therefore, eigen values of M are : 0, 0, 3

Correct answer is (b)

Example 42 : A 3 x 3 matrix M has $Tr[M] = 6, Tr[M^2] = 26, Tr[M^3] = 90$. Which of the following can be possible set of eigenvalues of M?

(a) {1,1,4} (b) {-1,0,7} (c) {-1,3,4} (d) {2,2,2}

Soln. $\lambda_1 + \lambda_2 + \lambda_3 = 6; \quad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 26; \ \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 90$

It is trivial to check that only $\{-1, 3, 4\}$ satisfies there three equation.

Correct answer is (c).

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Example 43 : The eigenvalues of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

(a) (1,4,9) (b) (0,7,7) (c) (0,1,13) (d) (0,0,14)

Soln. Eigenvalues of a n x n matrix having rows and columns as a scalar multiple of a particular row and column respectively, are trace,0,0,0,.....

Here, the second row and third row are twice and thrice of the first row and same is true for the columns.

So, the eigenvalues of the given matrix are 14, 0, 0

Correct answer is (d)

Example 44 : A 2 × 2 matrix 'A' has eigenvalues $e^{i\pi/5}$ and $e^{i\pi/6}$. The smallest value of 'n' such that $A^n = I$

(a) 20 (b) 30 (c) 60 (d) 120

Soln. A has eigenvalues $e^{i\frac{\pi}{5}}$ and $e^{i\frac{\pi}{6}}$

A^n has eigenvalues $e^{i\frac{n\pi}{5}}$ and $e^{i\frac{n\pi}{6}}$

So, $|A^n|$ = product of eigenvalues = $e^{i\frac{11n\pi}{30}}$

Since, $A^n = I \Rightarrow |A^n| = 1$

$\therefore \quad e^{i(11n\pi/30)} = 1 = e^{2m\pi i} \Rightarrow \frac{11n\pi}{30} = 2m\pi \Rightarrow \frac{n}{m} = \frac{60}{11}$

So, the smallest value of 'n' is 60.

Correct answer is (c)

Example 45 : A 3×3 matrix has eigenvalues 0, 2 + i and 2–i. Which one of the following statements is correct?

(a) the matrix is hermitian (b) the matrix is unitary
(c) the inverse of the matrix exists (d) the determinant of the matrix is zero

Soln. Determinant of matrix = product of the eigenvalues = 0

Correct option is (d)

Example 46 : A real traceless 4×4 unitary matrix has two eigen values –1 and 1. The other eigenvalues are
(a) zero and +2 (b) –1 and +1 (c) zero and +1 (d) +1 and +1

Soln. Sum of the eigenvalues = Trace of the matrix

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0 \Rightarrow 1 + (-1) + \lambda_3 + \lambda_4 = 0$

$\Rightarrow \lambda_3 + \lambda_4 = 0$

So, $\lambda_3 = +1, \lambda_4 = -1$

Correct option is (b)

Example 47 : A real traceless 4×4 unitary matrix has two eigen values –1 and 1. The other eigenvalues are
(a) zero and +2 (b) –1 and +1 (c) zero and +1 (d) +1 and +1

Soln. Sum of the eigenvalues = Trace of the matrix

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0 \Rightarrow 1 + (-1) + \lambda_3 + \lambda_4 = 0$

$\Rightarrow \lambda_3 + \lambda_4 = 0$

So, $\lambda_3 = +1, \lambda_4 = -1$

Correct option is (b)

Example 48 : Eigen value of the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i \\ 0 & 0 & 2i & 0 \end{pmatrix}$ are

(a) –2, –1, 1, 2 (b) –1, 1, 0, 2 (c) 1, 0, 2, 3 (d) –1, 1, 0, 3

Soln. Sum of the eigenvalues = Trace of the matrix = 0
Only option (a) satisfies this.
Correct option is (a)

Example 49 : One of the eigen values of the matrix $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is 5. The other two eigen values are

(a) 0 and 0 (b) 1 and 1 (c) 1 and -1 (d) -1 and -1

Soln. Sum of the eigenvalues = Trace of the matrix = 5

$\Rightarrow \quad \lambda_1 + \lambda_2 + \lambda_3 = 5$

$\Rightarrow \quad 5 + \lambda_2 + \lambda_3 = 5 \Rightarrow \lambda_2 + \lambda_3 = 0$

So, $\lambda_2 = 1$ and $\lambda_2 = -1$

Correct option is (c)

Example 50 : The normalized eigen vector corresponding to the eigen value 5 is:

(a) $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ (b) $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (c) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ (d) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Soln. Eigenvalue equation : $(A - \lambda I)X = 0$

$$\Rightarrow \quad \begin{bmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Putting $\lambda = 5$, we get

$$\Rightarrow \quad \begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \Rightarrow \quad -3x_1 + 3x_2 = 0 \text{ and } -4x_3 = 0$$

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Therefore, $X = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix}$, putting normalization condition $X^{\dagger}X = y \Rightarrow K = \frac{1}{\sqrt{2}}$

So, $X = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Correct option is (d)

Example 51 : The eigenvalues of a matrix are $i, -2i$ and $3i$. The matrix is

(a) unitary (b) anti-unitary (c) hermitian (d) anti-hermitian

Soln. Eigenvalues of an anti-hermitian matrix are either zero or purely imaginary

Correct option is (d)

Example 52 : The eigen values of the matrix $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are

(a) 5, 2, –2 (b) –5, –1, 1 (c) 5, 1, –1 (d) –5, 1, 1

Soln. Sum of the eigenvalues = Trace of the matrix

i.e. $\lambda_1 + \lambda_2 + \lambda_3 = 5$

Product of the eigenvalues = Determinant of the matrix

i.e. $\lambda_1\lambda_2\lambda_3 = -5$

Only option (c) satisfy both relations.

Correct option is (c)

Example 53 : A 3 × 3 matrix has elements such that its trace is 11 and its determinant is 36. The eigenvalues of the matrix are all known to be positive integers. The largest eigenvalue of the matrix is :

(a) 18 (b) 12 (c) 9 (d) 6

Soln. Trace of the matrix = $\lambda_1 + \lambda_2 + \lambda_3 = 11$

Determinant of the matrix = $\lambda_1\lambda_2\lambda_3 = 36$

Combination of λ_1, λ_2 and λ_3 that will satisfy both relation will be $\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 2$.

Correct option is (d)

Example 54 : The eigenvalues of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ are

(a) 0, 1, 1 (b) $0, -\sqrt{2}, \sqrt{2}$ (c) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$ (d) $\sqrt{2}, \sqrt{2}, 0$

Soln. Trace of the matrix = $\lambda_1 + \lambda_2 + \lambda_3 = 0$

Only option (b) satisfy this.

Correct option is (b)

Example 55 : The degenerate eigenvalues of the matrix $M = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}$ is:.....

(Your answer should be an integer)

Soln. Trace of the matrix = $\lambda_1 + \lambda_2 + \lambda_3 = 12$

Determinant of the matrix = $\lambda_1\lambda_2\lambda_3 = 50$

The combination $\lambda_1 = 5, \lambda_2 = 5, \lambda_3 = 2$ satisfies the above two relations.

Correct answer is (5)

2.3 Cayley-Hamilton Theorem:

Every square matrix satisfies its own characteristic equation or eigenvalue equation.

Example 56: $A=\begin{bmatrix}1 & 2\\ 4 & -6\end{bmatrix} \Rightarrow$ It's characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow\begin{vmatrix}1-\lambda & 2\\ 4 & -6-\lambda\end{vmatrix}=0\Rightarrow(\lambda+6)(\lambda-1)-8=0\Rightarrow\lambda^2+5\lambda-14=0$$

Therefore, Matrix A satisfies the above equation i.e. $A^2+5A-14I=0$

Calculation of the inverse of the given matrix:

Since, $A^2+5A=14I \Rightarrow A^{-1}A^2+5A^{-1}A=14A^{-1}I \Rightarrow A^{-1}=\frac{1}{14}(A+5I)$

Example 57: Let P be a n×n diagonalizable matrix which satisfies the equations

$$P^2=P, Tr(P)=n-1$$

Det (P) is :

(a) n (b) 0 (c) 1 (d) $n-1$

Soln: Using Cayley-hamilton theorem, $P^2-P=0 \Rightarrow\lambda^2-\lambda=0 \Rightarrow\lambda(\lambda-1)=0 \Rightarrow\lambda=0,1$

$\det(M)=\lambda_1.\lambda_2.....\lambda_n=0$ (Since, one of the eigenvalue is zero)

Example 58 : Let M be a 3×3 Hermitian matrix which satisfies the matrix equation

$$M^2-5M+6I=0$$

Where I refers to the identity matrix. Which of the following are possible eigenvalues of M

(a) {1, 2, 3} (b) {2, 2, 3} (c) {2, 3, 5} (d) 5, 5, 6}

Soln: According to Cayley-Hamilton theorem, we can write $\lambda^2-5\lambda+6=0\Rightarrow\lambda=2,3$

Correct option is (b)

Example 59: The eigenvalues of matrix $\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$ are :

(a) +1,+1,+1 and +1 (b) +1,+1,−1 and -1

(c) $0,0,\sqrt{2}$ and $-\sqrt{2}$ (d) $+\sqrt{2},-\sqrt{2},+1/\sqrt{2}$ and $-1/\sqrt{2}$

Soln: The given matrix A satisfies the relation $A^2=I$

According to Cayley-Hamilton theorem, we can write $\lambda^2=1\Rightarrow\lambda=\pm1$

Correct option is (b)

Example 60: The eigenvalues of the following matrix $A=\begin{bmatrix}3 & 2 & 9\\ 7 & 5 & 13\\ 6 & 17 & 19\end{bmatrix}$ are given by solving the cubic equation :

(a) $\lambda^3-27\lambda^2+167\lambda-285=0$ (b) $\lambda^3-27\lambda^2-122\lambda-313=0$

(c) $\lambda^3+27\lambda^2+167\lambda+285=0$ (d) $\lambda^3+23.23\lambda^2-158.3\lambda+313=0$

Soln. From Caley-Hamilton theorem, we know every square matrix satisfies it's own characteristic equation, i.e., $|A-\lambda I|=0$

Hence, $\det(A-\lambda I)=0$

$$\Rightarrow \det\left\{\begin{bmatrix}3 & 2 & 9\\ 7 & 5 & 13\\ 6 & 17 & 19\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}\right\}=0$$

$$\begin{vmatrix}3-\lambda & 2 & 9\\ 7 & 5-\lambda & 13\\ 6 & 17 & 19-\lambda\end{vmatrix}=0$$

On solving this, we get $\lambda^3-27\lambda^2-122\lambda-313=0$

Correct option is (b)

Example 61: The trace of a real 4 x 4 matrix U = exp(A), where $A=\begin{pmatrix}0 & 0 & 0 & \frac{\pi}{4}\\ 0 & 0 & -\frac{\pi}{4} & 0\\ 0 & \frac{\pi}{4} & 0 & 0\\ -\frac{\pi}{4} & 0 & 0 & 0\end{pmatrix}$ is

(a) $2\sqrt{2}$ (b) $\frac{\pi}{4}$ (c) 0 (d) 2 **[TIFR 2011]**

Example 62: Let $A=\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}$, then the eigenvalues of the matrix $B=A^4-3A^3+3A^2-2A+8I$, are

(a) $2+i, 2-i$ (b) $i, -i$ (c) $1+i, 1-i$ (d) None of these

Soln. Characteristic equation of matrix A, $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}1-\lambda & -1\\ 2 & 3-\lambda\end{vmatrix}=0 \Rightarrow \lambda^2-4\lambda+5$$

From Caley-Hamilton theorem matrix A will satisfy it's characteristic equation.

Hence, we have $A^2-4A+5=0$

Now, $B = A^4 - 3A^3 + 3A^2 - 2A + 8I$

$= (A^2 - 4A + 5I)(A^2 + A + 2I) + (A - 2I)$

$B = A - 2I$

$B = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

Hence, eigenvalues of B

$\Rightarrow \begin{vmatrix} -1-\alpha & -1 \\ 2 & 1-\alpha \end{vmatrix} = 0 \Rightarrow -(1+\alpha)(1-\alpha) + 2 = 0$

$\Rightarrow -1 + \alpha^2 + 2 = 0$

Hence, $\alpha^2 = 1 \Rightarrow \alpha = \pm i$.

Correct option is (b)

Example 63: For the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, A^{-1} is given by

(a) $A^2 = 2A$ (b) $A^2 + 2A + 3I$ (c) $A^2 - 2A - I$ (d) $A - 3I$

Soln. Characteristic equation of A, $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 2 & I-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(1 + \lambda^2 - 2\lambda - 1) + 1(2 - 1 + \lambda) = 0$

$\Rightarrow -\lambda^3 + 2\lambda^2 + 1 + \lambda = 0 \Rightarrow \lambda^3 - 2\lambda^2 - \lambda - 1 = 0$

From Caley-Hamilton theorem, $A^3 - 2A^2 - A - I = 0$ or $A^{-1}A^3 - 2A^{-1}A^2 - A^{-1}A - A^{-1}I = 0$

$\Rightarrow A^{-1} = A^2 - 2A - I$.

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Correct option is (c)

Example 64 : If a square matrix $A(3 \times 3)$ satisfies the equation $A^3 - 8A^2 + 17A - 10I = 0$ and it's eigenvalues are given by $\lambda_1, \lambda_2, \lambda_3$. Then, the value of $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$ is equal to _____.

Soln. $A^3 - 8A^2 + 17A - 10I = 0$

If $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of A, then they are the roots of this equation.

Hence, $\lambda_1 + \lambda_2 + \lambda_3 = 8$

$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 17$.

Correct answer is (17)

2.4 Rank of a matrix

A natural number 'r' is said to be the rank of the matrix A if it has the following properties:

(i) There exists at least one squre sub-matrix of the matrix A of order 'r' having non-zero determinant.

(ii) Every square sub-matrix of matrix A of order '$r + 1$' have zero determinant.

i.e. Rank of matrix ≡ order of the largest square sub-matrix of the given matrix having non-vanishing determinant.

Note: (i) The rank of the null matrix is zero.

(ii) The rank of any non-zero matrix is ≥ 1.

(iii) The rank of any non-singular matrix of order 'n' is 'n'.

(iv) The rank of a product of two matrices cannot exceed the rank of either matrix.

Example 65 : Find the rank of the matrix, (i) $A=\begin{bmatrix}3 & 2 & -1\\ 0 & 3 & 4\\ 5 & 1 & 2\end{bmatrix}$, (ii) $B=\begin{bmatrix}2 & 1 & -1\\ 0 & 3 & -2\\ 2 & 4 & -3\end{bmatrix}$

Soln: (i) $A=\begin{bmatrix}3 & 2 & -1\\ 0 & 3 & 4\\ 5 & 1 & 2\end{bmatrix} \Rightarrow |A| = 3(6-4)-2(0-20)+(-1)(0-15)=6+40+15=61\neq 0$; So, $\rho(A)=3$

(ii) $B=\begin{bmatrix}2 & 1 & -1\\ 0 & 3 & -2\\ 2 & 4 & -3\end{bmatrix} \Rightarrow |B| = 2(-9+8)-1(0+4)+(-1)(0-6)=-2-4+6=0$; So, $\rho(B)\neq 3$

Now, a 2×2 sub-matrix of B: $\begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}$, it's determinant $= 6$, So, $\rho(B)=2$

Example 66: The rank of matrix $\begin{bmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{bmatrix}$ (a, b, c, being all real), is 3 if

(a) $a=b=c$

(b) a, b, c are all different but $a+b+c=0$

(c) Two of the numbers a, b, c are equal but are different from the third

(d) a, b, c are all different and $a+b+c\neq 0$

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Soln: $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix}$ (applying $C'_3=C_3-C_1, C'_2=C_2-C_1$) $=\begin{vmatrix}1 & 0 & 0\\ a & b-a & c-a\\ a^3 & b^3-a^3 & c^3-a^3\end{vmatrix}$

$$=(b-a)(c-a)\begin{vmatrix}1 & 0 & 0\\ a & 1 & 1\\ a^3 & b^2+ab+a^2 & c^2+ca+a^2\end{vmatrix}=(b-a)(c-a)(b-c)(a+b+c)$$

The rank of the matrix is 3 if all a, b, c are different and $a+b+c\neq 0$

Correct option is (d)

Example 67: The rank of the matrix $\begin{pmatrix} a & -1 & 0 \\ 0 & a & -1 \\ -1 & 0 & a \end{pmatrix}$ is 2 for the value of 'a' equal to _____.

Soln. Since, given matrix has rank 2. Hence, determinant of 3×3 submatrix will be zero.

i.e., $\begin{vmatrix} a & -1 & 0 \\ 0 & a & -1 \\ -1 & 0 & a \end{vmatrix} = 0 \Rightarrow a = 1.$

Correct answer is (1)

2.5 Diagonalization of Matrix

Let, $\lambda_1, \lambda_2, \lambda_3 \ldots\ldots\ldots \lambda_n$ be 'n' distinct eigen values of a diagonalisable matrix A and $X_1, X_2 \ldots X_n$ be the corresponding linearly independent eigenvectors of the matrix A.

Diagonalized matrix D corresponding to A: $D = P^{-1} AP$

where matrix P is given by,

$$P = \begin{bmatrix} x_{11} & x_{12} & \cdots \\ x_{21} & x_{22} & \cdots \\ x_{31} & x_{32} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \quad \text{where } X_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}$$

Example 68: Diagonalize the matrix $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Soln: Eigenvalue equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \text{ i.e. } \lambda_1 = 1, \lambda_2 = -1$$

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Now the eigenvector corresponding to λ_1 is let X_1.

Therefore, $AX_1 = \lambda_1 X_1$

$$\Rightarrow \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} -ix_2 \\ ix_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow x_1 = -ix_2 \ \& \ x_2 = ix_1$$

Let $x_1 = k$, $x_2 = ik$ $\quad \therefore \quad X_1 = \begin{pmatrix} k \\ ik \end{pmatrix}$

Using the normalization condition i.e. $X_1^{\dagger} X_1 = 1$

$$\Rightarrow \quad \begin{pmatrix} k^* & -ik^* \end{pmatrix}\begin{pmatrix} k \\ ik \end{pmatrix} = 1 \quad \Rightarrow \quad |k|^2 + |k|^2 = 1 \quad \Rightarrow \quad k = \frac{1}{\sqrt{2}}$$

This gives $\quad X_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ +\frac{i}{\sqrt{2}} \end{pmatrix}$

Similarly, eigenvector corresponding to the eigenvalue $\lambda_2 = -1$ is $X_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$

Now, Diagonalizing matrix $P = (X_1 \ X_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$

This matrix P is the matrix which can diagonalize A and the diagonalized matrix will be

$$D = P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Altrenative Method:

(i) First find the eigen value of A i.e. $\lambda_1, \lambda_2 .... \lambda_n$

(ii) Now, if all eigenvalues distinct (no two eigenvalues are equal), then putting eigenvalues in place of principal diagonal elements and writting zero for all off diagonal elements, we will get D as

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & \cdots & \cdots \\ 0 & 0 & \ddots & \cdots \\ \vdots & \vdots & \vdots & \lambda_n \end{bmatrix}$$

Calculation of Power of a Matrix:

According to similarity tranformation, $D = P^{-1}AP \quad \Rightarrow A = PDP^{-1}$

$\Rightarrow A^K = PD^KP^{-1}$ (K is a positive integer)

Calculation of Exponential of a Matrix:

According to similarity tranformation, $D = P^{-1}AP \quad \Rightarrow A = PDP^{-1}$

$\Rightarrow (exp\ A) = P\ (exp\ D)\ P^{-1}$

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Calculation of Logarithm of a Matrix:

According to similarity tranformation, $D = P^{-1}AP \quad \Rightarrow A = PDP^{-1}$

$\Rightarrow (ln\ A) = P\ (ln\ D)\ P^{-1}$

Example 69: If $A = \begin{bmatrix} -2 & -1 & 0 \\ -2 & 2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$ is diagonalizable. 'S' is a matrix such that $S^{-1}AS = D_A$.

(D_A is diagonal matrix of A), then D_A is

(a) $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Soln. $A=\begin{bmatrix}-2 & -1 & 0\\ -2 & 2 & 0\\ 2 & 2 & 3\end{bmatrix}$

Hence, it's characteristic equation $|A-\lambda I|=0$

$$\begin{vmatrix}-2-\lambda & -1 & 0\\ -2 & 2-\lambda & 0\\ 2 & 2 & 3-\lambda\end{vmatrix}=0 \Rightarrow (1-\lambda)(3-\lambda)^2=0 \Rightarrow \lambda=1,3,3$$

Hence, $D_A=\begin{pmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{pmatrix}$.

Correct option is (d)

Example 70: If $A=\begin{pmatrix}-1 & 0 & 0\\ 1 & 2 & 0\\ 3 & 1 & 4\end{pmatrix}$ is a diagonalizable matrix and if exponential of A is expended as $e^A=\sum_{n=0}^{\infty}\frac{A^n}{n!}$, then the value of $\det(3e^A)$ is equal to

(a) e^5 (b) $27e^5$ (c) e^{-8} (d) $27e^{-8}$

Soln. We have $\det(e^A)=e^{Tr(A)}$

Therefore, $\det(3e^A)=3^3 e^{Tr(A)}=27e^5$.

Correct option is (b)

Example 71: Given a matrix $M=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$, which of the following represents $\cos(\pi M/6)$? **[JEST 2016]**

(a) $\frac{1}{2}\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ (b) $\frac{\sqrt{3}}{4}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$ (c) $\frac{\sqrt{3}}{4}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ (d) $\frac{1}{2}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}$

Soln: Given: $M=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$ and $N=\frac{\pi M}{6}=\begin{bmatrix}\pi/3 & \pi/6\\ \pi/6 & \pi/3\end{bmatrix}$

Eigenvalues of N are $\lambda_1=\frac{\pi}{6}, \lambda_2=\frac{\pi}{2}$

Eigenvectors of N are $X_1=\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}$ and $X_2=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$

Now, we can use similarly transformation to diagonalize N i.e.

$$D=P^{-1}NP$$

$$\Rightarrow \quad N=PDP^{-1} \Rightarrow \cos N=P(\cos D)P^{-1}$$

Here, $D=\begin{bmatrix}\pi/6 & 0\\ 0 & \pi/2\end{bmatrix}$, $P=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$, $P^{-1}=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$

$$\therefore \quad \cos N=\cos\left(\frac{\pi M}{6}\right)=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\cos\frac{\pi}{6} & 0\\ 0 & \cos\frac{\pi}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}=\frac{\sqrt{3}}{4}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Correct option is (b)

Example 72: Consider two matrices $A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $B=\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$. Now choose correct statement(s) :

(a) A and B both are diagonalizable. (b) B is diagonalizable but not A.

(c) $Tr(\log B)=\log 2$ (d) $\det(e^A)=e^2$

Soln. A and B both are upper triangular matrices.
Hence, eigenvalues of A are 1, 1 and eigenvalues of B are 1, 2.
Since, eigenvalues of B are distinct, it is diagonalizable.
Also, from similarly transformation $P^{-1}BP=D_B \Rightarrow B=PD_BP^{-1}$

Here, $D_B=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$. Hence, $\log B=P\log D_BP^{-1}$ or $Tr(\log B)=Tr(\log D_B)$.

Here, $\log D_B=\begin{pmatrix}\log 1 & 0\\ 0 & \log 2\end{pmatrix}$. Therefore, $Tr(\log D_B)=\log 1+\log 2=\log 2$.

Also, $\det(e^A)=e^{Tr(A)}=e^2$.

Correct options are (b), (c) and (d)

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Example 73: If $A_{n\times n}$ is symmetric matrix with real entries and all eigenvalues distinct, then which of the following is correct?

(a) The eigenvalues are real.
(b) Eigenvectors corresponding to distinct eigenvalues are orthogonal.
(c) A satisfies $A=PDP^T$ (Here D is a square matrix whose columns are the eigenvectors of the eigenvalues of A and D is a diagonal matrix).
(d) The eigenvalues are complex.

Soln. Symmetric matrix with real entries will always have real eigenvalues.
Also, eigenvectors corresponding to different eigenvalues are always orthogonal.
Hence, P is also an orthogonal matrix.

i.e., $P^TP=I$ or $P^T=P^{-1}$

Now, $A=PDP^T$

$A=PDP^{-1}$ or $P^{-1}AP=D$

(Similarly transformation for diagonalization of A).

Correct options are (a), (b) and (c)

Example 74: If $A=\begin{pmatrix}\pi/2 & 4\\ 0 & 3\pi\end{pmatrix}$ and $B=\sin\left(\frac{A}{3}\right)$. If $\sin A=\sum_{n=0}^{\infty}(-1)^{n+2}\frac{A^{2n+1}}{(2n+1)!}$, then trace of B is _____.

Soln. Here, $\frac{A}{3}=\begin{pmatrix}\pi/6 & 4/3\\ 0 & \pi\end{pmatrix}=C$ (let)

Eigenvalues of $\left(\frac{A}{3}\right)$ are $\frac{\pi}{6}$, π, hence it is diagonalizable.

(Since, eigenvalues are distinct, so corresponding eigenvectors will be linearly independent).

Therefore, $C=P\,D_C\,P^{-1}$ or $\sin C=P\sin D_C\,P^{-1}$

Here, $D_C=\begin{pmatrix}\pi/6 & 0\\ 0 & \pi\end{pmatrix}\Rightarrow \sin D_C=\begin{pmatrix}\sin\frac{\pi}{6} & 0\\ 0 & \sin\pi\end{pmatrix}$

Therefore, $B=\sin C=\sin\frac{A}{3}=P\sin D_C P^{-1}$

Therefore, trace of $B=$ trace of $\sin D_C=\sin\frac{\pi}{6}+\sin\pi=\frac{1}{2}$.

Correct answer is (0.5)

Example 75: Consider the matrix $M=\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$

The exponential of M simplifies (I is the 3 x 3 identity matrix)

(a) $e^M=I+\left(\frac{e^3-1}{3}\right)M$ (b) $e^M=I+M+\frac{M^2}{2!}$ (c) $e^M=I+3^3M$ (d) $e^M=(e-1)M$

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Soln. $e^M=I+M+\frac{M^2}{2!}+\frac{M^3}{3!}+.....$

Now, $M^2=\begin{pmatrix}3 & 3 & 3\\ 3 & 3 & 3\\ 3 & 3 & 3\end{pmatrix}=3M$, $M^3=\begin{pmatrix}9 & 9 & 9\\ 9 & 9 & 9\\ 9 & 9 & 9\end{pmatrix}=3^2M$ and so on

Therefore, $e^M=I+M+\frac{3M}{2!}+\frac{3^2M}{3!}+.....=I+\frac{M}{3}\left(3+\frac{3^2}{2!}+\frac{3^3}{3!}+.....\right)=I+\frac{M}{3}\left(e^3-1\right)$

Correct answer: (a)

Example 76: If two matrices A and B can be diagonalized simultaneously, which of the following is true?

(a) $A^2B=B^2A$ (b) $A^2B^2=B^2A$ (c) $AB=BA$ (d) $AB^2AB=BABA^2$

Soln. The diagonal matrices D_A and D_B corresponding respectively, will commute with each other.

$D_AD_B=D_BD_A$

$\Rightarrow P^{-1}APP^{-1}BP = P^{-1}BPP^{-1}AP$ [same matrix is used for A and B]

$\Rightarrow P^{-1}ABP = P^{-1}BAP \Rightarrow AB = BA$

Correct option is (c)

Example 77: Two matrices A and B are said to be similar if $B = P^{-1}AP$ for some invertible matrix P. Which of the following statements is **NOT TRUE** ?

(a) Det A = Det B (b) Trace of A = Trace of B

(c) A and B have the same eigenvectors (d) A and B have the same eigenvalues

Soln. Two similar matrices A and B have same trace, determinant eigenvalues and rank.

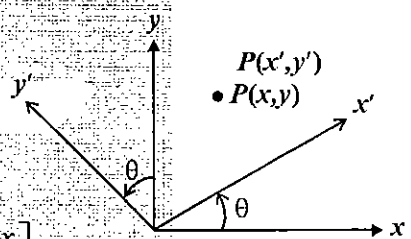
Correct option is (c)

2.6 Applications

Rotation Matrix:

Let (x, y, z) be a cartesian coordinate system in a three dimension space. Let $\vec{r} = x\hat{i} + y\hat{j}$ be a vector in x-y plane. If we consider a rotation of coordinate system about z-axis through an angle θ in anticlockwise sense and indicate the new axis by (x', y'), then the components of the same vector $\vec{r} = x'\hat{i}' + y'\hat{j}'$ relative to the new system may be expressed as

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$



Then equations in matrix form can be expressed as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = R_z(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$$

where $R_z(\theta)$ is the transformation matrix corresponding to the rotation in x-y plane about z-axis.

Note: (i) $R_z(\theta)$ is orthogonal matrix.

(ii) Determinant of $R_z(\theta)$ is 1.

(iii) Eigenvalues of $R_z(\theta)$ are $e^{i\theta}$ and $e^{-i\theta}$.

Let us consider a vector in space i.e. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Here, three types of rotation will be possible:

CASE I: Rotation about z-axis:

$$\text{Rotation matrix. } R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

CASE II: Rotation about x-axis:

$$\text{Rotation matrix. } R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

CASE III: Rotation about y-axis:

$$\text{Rotation matrix. } R_y(\theta)=\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}$$

Example 78: Consider a vector $\vec{P}=\hat{i}+2\hat{j}+\hat{k}$ in the co-ordinate system $(\hat{i},\hat{j},\hat{k})$. The axes are rotated in the clockwise direction about the z axis by an angle 45°. The vector $\vec{P}$ in the rotated co-ordinate system will be

(a) $-\frac{1}{\sqrt{2}}\hat{i}'+\frac{3}{\sqrt{2}}\hat{j}'+\hat{k}'$ (b) $\frac{1}{\sqrt{2}}\hat{i}'-\frac{3}{\sqrt{2}}\hat{j}'+\hat{k}'$ (c) $\frac{3}{\sqrt{2}}\hat{i}'+\frac{1}{\sqrt{2}}\hat{j}'+\hat{k}'$ (d) $\frac{3}{\sqrt{2}}\hat{i}'-\frac{1}{\sqrt{2}}\hat{j}'+\hat{k}'$

Soln: $$\begin{bmatrix}x'\\ y'\\ z'\end{bmatrix}=\begin{bmatrix}\cos45^0 & -\sin45^0 & 0\\ \sin45^0 & \cos45^0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\cos45^0 & -\sin45^0 & 0\\ \sin45^0 & \cos45^0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}=\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{3}{\sqrt{2}}\\ 1\end{bmatrix}$$

Correct Option is (a)

Example 79: Under a rotation by an angle θ about the x-axis, the vector that remains invariant is

(a) $2\hat{x}-\hat{y}+\hat{z}$ (b) $3\hat{z}$ (c) $5\sqrt{2}\hat{x}$ (d) $\sqrt{3}\hat{y}$

Soln: Since, rotation take place about x-axis.

$$\text{Rotation matrix. } R_x(\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}$$

So, x-coordinate will remain invariant.

Correct Option is (c)

Example 80: Consider a vector $\vec{p}=2\hat{i}+3\hat{j}+2\hat{k}$ in the coordinate system $(\hat{i},\hat{j},\hat{k})$. The axes are rotated anti-clockwise about the Y axis by an angle of 60°. The vector $\vec{p}$ in the rotated coordinate sytem $(\hat{i}',\hat{j}',\hat{k}')$ is

(a) $\left(1-\sqrt{3}\right)\hat{i}'+3\hat{j}'+\left(1+\sqrt{3}\right)\hat{k}'$ (b) $\left(1+\sqrt{3}\right)\hat{i}'+3\hat{j}'+\left(1-\sqrt{3}\right)\hat{k}'$

(c) $\left(1-\sqrt{3}\right)\hat{i}'+\left(3+\sqrt{3}\right)\hat{j}'+2\hat{k}'$ (d) $\left(1-\sqrt{3}\right)\hat{i}'+\left(3-\sqrt{3}\right)\hat{j}'+2\hat{k}'$

Soln. The rotation matrix corresponding to the anti-clockwise rotation about y-axis by an angle θ will be

$$R_y(\theta)=\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos 60^\circ & 0 & -\sin 60^\circ \\ 0 & 1 & 0 \\ \sin 60^\circ & 0 & \cos 60^\circ \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \left(1-\sqrt{3}\right) \\ 3 \\ \left(1+\sqrt{3}\right) \end{bmatrix}$$

Correct option is (a)

Existance and uniqueness of the solution of system of Simultaneous Linear Non-Homogeneous Equations:

Consistent Equations: They have one or more solutions.

Example: (a) Unique Solution: $x+2y=4; 3x+2y=2$

(b) Infinite Solution: $x+2y=4; 3x+6y=12$

Inconsistent Equations: They have no solution.

Example: $x+2y=4; 3x+6y=5$

Suppose a system of 'n' simultaneous linear equations containing 'n' unknown variables as following:

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots$$
$$\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots\ldots + a_{nn}x_n = b_n$$

where co-efficient matrix of the system $= A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & \ldots\ldots & a_{n1} \\ a_{21} & a_{22} & \ldots\ldots & \ldots\ldots & a_{n2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & \ldots\ldots & a_{nn} \end{bmatrix}$

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and Augamented matrix of the system $= C = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & \ldots\ldots & a_{n1} & b_1 \\ a_{21} & a_{22} & \ldots\ldots & \ldots\ldots & a_{n2} & b_2 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & \ldots\ldots & a_{nn} & b_n \end{bmatrix}$

(1) If rank of A = rank of C, then the equations are consistent.

(a) If rank of A = rank of C = number of unknown variables, then there will be unique solution.

(b) If rank of A = rank of C < number of unknown variables, then there will be infinite solution.

(2) If rank of A $\neq$ rank of C, then the equations are inconsistent and have no solutions.

Question 1. The matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ **[TIFR 2010]**

can be relateed by a similarity transformation to the matrix.

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Soln: Since, the given matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, and the matrices given in options are related by similarity transformation. Hence, their trace and determinant would be same.

Correct option is (d)

Question 2. Consider the matrix, $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$

A 3-dimensional basis formed by eigenvectors of M is **[TIFR 2011]**

(a) $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ (d) $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

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Soln: Eigenvalue equation is $(M - \lambda I) = 0$

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & -1 & -\lambda \end{pmatrix} = 0 \; ; \; (1-\lambda)(\lambda^2 - 1) = 0 \; ; \; \lambda = 1, 1, -1$$

Now, eigenvectors of M.

For $\lambda = 1$; $(M - \lambda I)X = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \; ; \; -y - z = 0 \Rightarrow y = -z$$

x can have any value, if $z = k_1$, $y = -k_1$

Therefore, $X_{\lambda=1}=\begin{pmatrix} x \\ -k_1 \\ k_1 \end{pmatrix}$

For $\lambda=-1$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0 \;;\; 2x=0 \;\boxed{x=0}\; y-z=0 \;\boxed{y=z}$$

Hence, $X_{\lambda=-1}=\begin{pmatrix} 0 \\ k_2 \\ k_2 \end{pmatrix}$, which is satisfied by option (b) only.

Correct option is (b)

Question 3. The trace of the real 4×4 matrix $U=\exp(A)$, where $A=\begin{pmatrix} 0 & 0 & 0 & \pi/4 \\ 0 & 0 & -\pi/4 & 0 \\ 0 & \pi/4 & 0 & 0 \\ -\pi/4 & 0 & 0 & 0 \end{pmatrix}$

is equal to **[TIFR 2011]**

(a) $2\sqrt{2}$ (b) $\pi/4$ (c) $\exp(i\varphi)$ for $\varphi=0,\pi$ (d) zero

Soln: $A^2=-\frac{\pi^2}{16}I;\; A^3=A^2.A=-\frac{\pi^2}{16}A;\; A^4=\left(A^2\right)^2=\frac{\pi^4}{256}I$; and so on .....

$$\exp(A)=I+A+\frac{A^2}{2!}+\frac{A^3}{3!}+\ldots\ldots=I+A-\frac{1}{2!}\frac{\pi^2}{16}I-\frac{1}{3!}\frac{\pi^2}{16}A+\frac{1}{4!}\frac{\pi^4}{256}I+\ldots\ldots$$

$$Tr\left[\exp(A)\right]=Tr(I)+Tr(A)-Tr\left(\frac{1}{2!}\frac{\pi^2}{16}I\right)-Tr\left(\frac{1}{3!}\frac{\pi^2}{16}A\right)+Tr\left(\frac{1}{4!}\frac{\pi^4}{256}I\right)+\ldots\ldots$$

$$=4+0-4.\frac{1}{2!}\frac{\pi^2}{16}-0+4.\frac{1}{4!}\frac{\pi^4}{256}+\ldots\ldots=4\left[1-\frac{1}{2!}\frac{\pi^2}{16}+\frac{1}{4!}\frac{\pi^4}{256}-\ldots\ldots\right]=4\cos\left(\frac{\pi}{4}\right)=2\sqrt{2}$$

Question 4. Two different 2×2 matrices A and B are found to have the same eigenvalues. It is then correct to state that $A=SBS^{-1}$ where S can be a **[TIFR 2012]**

(a) traceless 2×2 matrix (b) Hermitian 2×2 matrix

(c) unitary 2×2 matrix (d) arbitrary 2×2 matrix

Soln: $A=S\,B\,S^{-1}$; since, matrix 'S' is formed by the eigenvectors of A which are orthogonal and normalized. Hence, 'S' would be an unitary matrix.

Correct option is (c)

Question 5. The product MN of two Hermitian matrices M and N is anti-Hermitian. It follows that **[TIFR 2014]**

(a) $\{M,N\}=0$ (b) $[M,N]=0$ (c) $M^{\dagger}=N$ (d) $M^{\dagger}=N^{-1}$

Soln: Since, M and N are Hermitian matrix. Therefore, $M^{+}=M$ and $N^{+}=N$.

Also, given $(MN)^{+}=-MN \Rightarrow N^{+}M^{+}=-MN$; $NM+MN=0 \Rightarrow \{M,N\}=0$.

Correct option is (a)

Question 6. If the eigenvalues of a symmetric 3×3 matrix A are 0, 1, 3 and the corresponding eigenvectors can be written as **[TIFR 2016]**

$$\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

respectively, then the matrix A^4 is

(a) $\begin{bmatrix}41&-81&40\\-81&0&-81\\40&-81&41\end{bmatrix}$ (b) $\begin{bmatrix}-82&-81&79\\-81&81&-81\\79&-81&83\end{bmatrix}$ (c) $\begin{bmatrix}14&-27&13\\-27&54&-27\\13&-27&14\end{bmatrix}$ (d) $\begin{bmatrix}14&-13&27\\-13&54&-13\\27&-13&14\end{bmatrix}$

Soln: Since, eigenvalues of A are 0, 1, 3. Hence, eigenvalues of A^4 are 0, 1, 81

Therefore, Trace $(A^4)=82 \Rightarrow \det(A^4)=0$.

Check Trace and determinant of all options.

Correct option is (c)

Question 7. Denote the commutator of two matrices A and B by $[A,B]=AB-BA$ and the anti-commutator by $\{A,B\}=AB+BA$. If $\{A,B\}=0$, we can write $[A,BC]=$ **[TIFR 2017]**

(a) $-B[A,C]$ (b) $B\{A,C\}$ (c) $-B\{A,C\}$ (d) $[A,C]B$

Soln: Since, $\{AB\}=0 \Rightarrow AB+BA=0$ ... (1)

Now, $[A,BC]=0$

$\Rightarrow B[A,C]+[A,B]C=0 \Rightarrow B(AC-CA)+(AB-BA)C=0 \Rightarrow BAC-BCA+ABC-BAC=0$

From Eq. (1) ; $AB=-BA$

Therefore, $-BCA-BAC=0 \Rightarrow -B(AC+CA)=0$

Therefore, $-B\{A,C\}=0$

Correct option is (c)

Question 8. The matrix $\begin{pmatrix}100\sqrt{2}&x&0\\-x&0&-x\\0&x&100\sqrt{2}\end{pmatrix}$, where $x>0$, is known to have two equal eigenvalues.

Find the value of x. **[TIFR 2017]**

Soln: $A=\begin{pmatrix} 100\sqrt{2} & x & 0 \\ -x & 0 & -x \\ 0 & x & 100\sqrt{2} \end{pmatrix}$

Eigenvalue equation is $(A-\lambda I)=0$, $A=\begin{pmatrix} 100\sqrt{2}-\lambda & x & 0 \\ -x & -\lambda & -x \\ 0 & x & 100\sqrt{2}-\lambda \end{pmatrix}=0$

$\left(100\sqrt{2}-\lambda\right)\left(-\lambda\left(100\sqrt{2}-\lambda\right)+x^2\right)-x\left(-x\left(100\sqrt{2}-\lambda\right)-0\right)=0$

$\left(100\sqrt{2}-\lambda\right)\left\{2x^2+\lambda^2-100\sqrt{2}\lambda\right\}=0$; $\left(100\sqrt{2}-\lambda\right)\left\{\lambda^2-100\sqrt{2}\lambda+2x^2\right\}=0$

Let us suppose quadratic terms in 2nd bracket will give equal roots or eigenvalues.

Hence, $\left(100\sqrt{2}\right)^2-4\times1\times2x^2=0\Rightarrow x=50$.

Correct answer is (50)

Question 9. A unitary matrix U is expanded in terms of a Hermitian matrix H, such that $U=e^{i\pi H/2}$. If we know that

$$H=\begin{pmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

then U must be **[TIFR 2017]**

(a) $\begin{pmatrix} i & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & i & \frac{1}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & i \end{pmatrix}$ (b) $\begin{pmatrix} \frac{i}{2} & 0 & \frac{i\sqrt{3}}{2} \\ 0 & i & 0 \\ \frac{i\sqrt{3}}{2} & 0 & -\frac{i}{2} \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 2 & 0 \\ \sqrt{3} & 0 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 2i & 1 & \frac{\sqrt{3}}{2} \\ 1 & 2i & 0 \\ \frac{\sqrt{3}}{2} & 0 & 2i \end{pmatrix}$

Soln: Since, U is an unitary matrix, hence just check the normalization condition for each rows and orthogonalization condition for two rows.

Correct option is (b)

Question 10. If a 2×2 matrix M is given by, $M=\begin{pmatrix} 1 & (1-i)/\sqrt{2} \\ (1+i)/\sqrt{2} & 0 \end{pmatrix}$

Then Det(expM) = **[TIFR 2018]**

(a) e (b) e^2 (c) $2i\sin\sqrt{2}$ (d) $\exp\left(-2\sqrt{2}\right)$

Soln: We know that,

$\det\left(e^M\right)=e^{Tr(M)}=e^{1+0}=e$

Correct option is (a)

JEST PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. For an $N \times N$ matrix consisting of all ones, **[JEST 2012]**

(a) all eigenvalues = 1 (b) all eigenvalues = 0

(c) the eigenvalues are 1, 2, ..., N (d) one eigenvalue = N, the others = 0

Soln: $A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$; Eigenvalues of A will be trace (A), 0, 0,... $(n-1)$ times. i.e., n, 0, 0, 0,... $(n-1)$ times.

Correct option is (d)

Question 2. The coordinate transformation

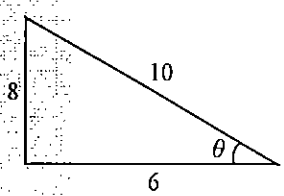
$$x' = 0.8x + 0.6y, \qquad y' = 0.6x - 0.8y$$

represents **[JEST 2013]**

(a) a translation (b) a proper rotation (c) a reflection (d) none of the above.

Soln: $x' = 0.8x + 0.6y,\ y' = 0.6x - 0.8y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



Compare this with rotation matrix $R_z(\theta)$ about z-axis.

$$R_z(\theta) = \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}, \text{ if } \sin\theta = 0.8 = \frac{8}{10}.$$

Therefore, $\cos\theta = \frac{6}{10} = 0.6$

Correct option is (b)

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Question 3. Given a matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, which of the following represents $\cos(\pi M/6)$?

[JEST 2016]

(a) $\frac{1}{2}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ (b) $\frac{\sqrt{3}}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ (c) $\frac{\sqrt{3}}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (d) $\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

Soln: Given: $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $N = \frac{\pi M}{6} = \begin{bmatrix} \pi/3 & \pi/6 \\ \pi/6 & \pi/3 \end{bmatrix}$

Eigenvalues of N are $\lambda_1 = \frac{\pi}{6}, \lambda_2 = \frac{\pi}{2}$

Eigenvectors of N are $X_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Now, we can use similarly transformation to diagonalize N i.e.

$D = P^{-1}NP \Rightarrow N = PDP^{-1} \Rightarrow \cos N = P(\cos D)P^{-1}$

Here, $D = \begin{bmatrix} \pi/6 & 0 \\ 0 & \pi/2 \end{bmatrix}$, $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$\therefore \quad \cos N = \cos\left(\frac{\pi M}{6}\right) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{6} & 0 \\ 0 & \cos\frac{\pi}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{\sqrt{3}}{4} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Correct option is (b)

Question 4. Let $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix}$ and $M = \begin{bmatrix} 10 & 3i \\ -3i & 2 \end{bmatrix}$. Similarity transformation from M to Λ can be performed by

[JEST 2017]

(a) $\frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix}$ (b) $\frac{1}{\sqrt{9}}\begin{pmatrix} 1 & -3i \\ 3i & 11 \end{pmatrix}$ (c) $\frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3i \\ -3i & 11 \end{pmatrix}$ (d) $\frac{1}{\sqrt{9}}\begin{pmatrix} 1 & 3i \\ -3i & 1 \end{pmatrix}$

Soln: $A = \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix}$ and $M = \begin{bmatrix} 10 & 3i \\ -3i & 2 \end{bmatrix}$

Since, 'M' and 'A' are related by similarly transformation. Hence, eigenvalues of M are 1 and 11.

Now, eigenvector corresponding to $\lambda = 1$

$$\begin{pmatrix} 10-\lambda & 3i \\ -3i & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 9 & 3i \\ -3i & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$9x + 3iy = 0$ ... (1)

$-3ix + y = 0$ ... (2)

Eq. (1) and Eq. (2) both are same.

$y = 3ix$, if $x = k$; $y = 3ik$

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Therefore, $X_{\lambda=1} = \begin{pmatrix} k \\ 3ik \end{pmatrix} = \begin{pmatrix} 1 \\ 3i \end{pmatrix}$ {if $k = 1$} and eigenvector corresponding to $\lambda = 11$

$$\begin{pmatrix} -1 & 3i \\ -3i & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \; ; \; -x + 3iy = 0 \; ; \; x = 3iy$$

Therefore, $X_{\lambda=11} = \begin{pmatrix} 3i \\ 1 \end{pmatrix}$

Therefore, similarly transformation from M to A will be performed by the matrix.

$\Rightarrow \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix}$ {normalized matrix}.

Correct option is (a)

JNU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The eigenvalues of the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ are **[JNU 2009]**

(a) $-1, -1$ and 2 (b) -1, 1 and 2 (c) $-1, -2$ and 2 (d) -1, 2 and 2

Soln: Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, then $\text{Tr}(A) = 0$ and $\det(A) = 2$.

Correct option is (a)

Question 2. The eigenvalues of matrix $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ are **[JNU 2009]**

(a) $+1, +1, +1$ and $+1$ (b) $+1, +1, -1$ and -1

(c) $0, 0, \sqrt{2}$ and $-\sqrt{2}$ (d) $+\sqrt{2}, -\sqrt{2}, +1/\sqrt{2}$ and $-1/\sqrt{2}$

Soln: Let $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Therefore, $A^2 = 1$ or eigenvalue equation is $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$, other eigenvalues will be repeated.

Correct option is (b)

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Question 3. You are given the following transformation of the coordinates (x,y) of a point in the two-dimensional plane :

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Which of the following statements describes the transformation ? **[JNU 2011]**

(a) Rotation by an angle of 45° about the z-axis followed by translation by one unit along both x-and y-axes

(b) Rotation by an angle of 45° about the z-axis followed by an expansion of both the x-and y-coordinates by a factor of $\sqrt{2}$

(c) Translation by one unit along both x- and y-axes followed by an expansion of both the x-and y-coordinates by a factor of $\sqrt{2}$

(d) Only a rotation by an angle 45° about the z-axis.

Soln: $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$; if axes is rotated about z-axis clockwise, then $R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, if $\theta = 45°$,

then $R_z(\theta)=\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$.

If both the axes are expanded by $\sqrt{2}$ times, then $R_z(\theta)=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$.

Correct option is (b)

Question 4. For which of the following matrices both the eigenvalues are positive? **[JNU 2011]**

(a) $\begin{pmatrix}-7 & 2\\ 2 & -2\end{pmatrix}$ (b) $\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ (c) $\begin{pmatrix}4 & -3\\ -3 & 5\end{pmatrix}$ (d) $\begin{pmatrix}-1 & 4\\ 4 & -1\end{pmatrix}$

Soln: Matrix whose trace and determinant both are positive will have both eigenvalues positive.

Correct option is (c)

Question 5. A homogeneous linear transformation takes the point (1, 1) in the xy-plane to the point (3, 3) and keeps the point (1, −1) fixed (i.e., it remains (1, −1) after the transformation). The matrix corresponding to this transformtion is **[JNU 2012]**

(a) $\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ (b) $\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}$ (c) $\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$ (d) $\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$

Soln: Let the transformation matrix is $T=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

Hence, according to question,

$$\begin{pmatrix}3\\ 3\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$a+b=3$ ... (1)

$c+d=3$ ... (2)

And $\begin{pmatrix}1\\ -1\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}$

$a-b=1$ ... (3)

$-c+d=1$ ... (4)

From equation (1) and (3), a = 2, b = 1 and from (2) and (4) d = 2, c = 1

$$\boxed{T=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}}$$

Correct option is (c)

Question 6. Let $\lambda_i\,(i=1,2,3)$ be the eigenvalues of the matrix. The sum $\sum_{i=1}^{3}\lambda_i^2$ is equal to

$$\begin{bmatrix}2 & -1 & -3\\ -1 & 1 & 2\\ -3 & 2 & 3\end{bmatrix}$$

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[JNU 2013]

(a) 14 (b) 42 (c) 6 (d) 0

Soln: Let $A=\begin{bmatrix}2 & -1 & -3\\ -1 & 1 & 2\\ -3 & 2 & 3\end{bmatrix}$. If $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of A, then $\lambda_1^2, \lambda_2^2, \lambda_3^2$ will be eigenvalues of A^2.

Here, $A^2=\begin{bmatrix}2 & -1 & -3\\ -1 & 1 & 2\\ -3 & 2 & 3\end{bmatrix}\begin{bmatrix}2 & -1 & -3\\ -1 & 1 & 2\\ -3 & 2 & 3\end{bmatrix}=\begin{bmatrix}14 & -9 & -19\\ -9 & 6 & 11\\ -17 & 11 & 22\end{bmatrix}$.

Now, $\sum_{i=1,2,3}\lambda_1^2+\lambda_2^2+\lambda_3^2=$ trace of $A^2=14+6+22=42$.

Correct option is (b)

Question 7. The exponential of a 2×2 matrix A is defined by the power series expansion $\exp(A)=\sum_{n=0}^{\infty}\frac{1}{n!}A^n$, A^0 being the 2×2 identify matrix. If $S=\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$ and $SAS^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, then the determinant of the matrix e^A is [JNU 2014]

(a) 1 (b) e^2 (c) $\frac{1}{e}$ (d) $e+\frac{1}{e}$ (e) -1

Soln: $S=\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$ and $SAS^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=D_A$ (let)

Hence, eigenvalues of A are 1, 1

Therefore, $S^{-1}D_AS$ or $e^A=S^{-1}e^{D_A}S$. Here, $e^{D_A}=\begin{pmatrix}e & 0\\ 0 & e\end{pmatrix}$

Therefore, $\det(e^A)=\det(e^{D_A})=e\times e=e^2$.

Correct option is (b)

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Question 8. The trace and the determinant of a 3×3 matrix A satisfy Tr A = 2 det A = 2. Further, the sum of two of the eigenvalues of A is equal to the third eigenvalue. Then the trace and the determination of the matrix A^2 are, respectively [JNU 2015]

(a) 2 and 1 (b) 0 and 1 (c) 4 and 1 (d) 4 and 2 (d) 4 and 4

Soln: Let $\lambda_1, \lambda_2, \lambda_3$ are three eigenvalues of A, then trace $(A)=\lambda_1+\lambda_2+\lambda_3$

$$\lambda_1+\lambda_2+\lambda_3=2 \quad \ldots(1)$$

and $\det(A)=\lambda_1\lambda_2\lambda_3$

$$\lambda_1\lambda_2\lambda_3=2 \quad \ldots(2)$$

Also, $\lambda_1+\lambda_2=\lambda_3$ ... (3)

Solving Eqs (1), (2) and (3)

$\lambda_3=1$; $\lambda_1+\lambda_2=1$; $\lambda_1\lambda_2=2$

No options are matched.

DU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. If the trace and the determinant of a 2×2 matrix are both zero; then, **[DU 2014]**

(a) Only one of eigenvalues needs to be zero (b) Both the eigenvalues need to be zero

(c) Both the eigenvalues are non-zero but finite (d) At least one of the eigenvalues should be infinite

Soln: Since, both trace and determinant of 2×2 matrix is zero, hence both eigenvalues must be zero.

Correct option is (b)

Question 2. The matrix given by: **[DU 2014]**

$$\begin{pmatrix} 3 & 2i & 1 \\ -2i & 4 & 1+2i \\ 1 & 1-2i & 5 \end{pmatrix}$$

(a) Has one real and one complex eigenvalues (b) Has two real and one complex eigenvalues

(c) Has three real eigenvalues (d) Has three complex eigenvalues

Soln: Given matrix is Hermitian in nature. Hence, it's all eigenvalues are real.

Correct option is (c)

Question 3. The trace of a 4×4 matrix is 8 and its determinant is -24. If two of its eigenvalues are 3 and 4, the other two eigenvalues are: **[DU 2014]**

(a) -2 and 1 (b) 2 and -1 (c) 2 and 1 (d) -2 and -1

Soln: Let eigenvalues of 4×4 matrix is $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, then given

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 8 \quad \ldots (1)$$

and $\lambda_1 \lambda_2 \lambda_3 \lambda_4 = -24 \quad \ldots (2)$

Also, $\lambda_1 = 3, \lambda_2 = 4 \quad \ldots (3)$

Hence, another eigenvalues must be 2 and -1.

Correct option is (b)

Question 4. Which of the following statements is true for a square matrix A?

(a) If $A^2 = 0$, it necessarily implies that $A = 0$ **[DU 2015]**

(b) If A is real and orthogonal, its eigenvalues will always be real

(c) If A is hermitian, its diagonal entries are always real

(d) If A is anti-hermitian (skew-hermitian), its diagonal entries are always zero

Soln: **Correct option is (c)**

Question 4. Under a rotation by an angle θ about the x-axis, the vector that remains invariant is

(a) $2\hat{x} - \hat{y} + \hat{z}$ (b) $3\hat{z}$ (c) $5\sqrt{2}\hat{x}$ (d) $\sqrt{3}\hat{y}$ **[DU 2015]**

Soln: Since, rotation is about x-axis, hence vector $5\sqrt{2}\,\hat{x}$ will remain invariant.

Correct option is (c)

Question 6. Consider the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$$

If B = $2e^A$, the determinant of B is **[DU 2016]**

(a) $4e^7$ (b) $4\left(e^7-e^4\right)$ (c) e^7 (d) e^7-e^4

Soln: $A=\begin{pmatrix}2 & 2\\ 2 & 5\end{pmatrix}$; $B=2e^A$

Therefore, $\det B=\det\left(2e^A\right)=2^2\det\left(e^A\right)$

$=4e^{\text{tra}\,A}$ $\left(\because \det e^A=e^{\text{tra}\,A}\right)$

$=4e^7$.

Correct option is (a)

Question 7. If $A=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ and I is a 2 × 2 identity matrix, then e^A is : **[DU 2017]**

(a) $I+\left(\frac{e^2-1}{2}\right)A$ (b) $I+\left(\frac{e-1}{2}\right)A$ (c) $I+\left(\frac{e^2+1}{2}\right)A$ (d) $I+\left(\frac{e+1}{2}\right)A$

Soln: $A=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$; $A^2=2A$, $A^3=A^2\cdot A=2A\cdot A=4A$, $A^4=A^2\cdot A^2=8A$ and so on ....

$$e^A=I+A+\frac{A^2}{2!}+\frac{A^3}{3!}+\frac{A^4}{4!}+\cdots$$

$$=I+A+\frac{2A}{2!}+\frac{4A}{3!}+\frac{8A}{4!}+\cdots$$

$$=I+A\left(I+\frac{2}{2!}+\frac{4}{3!}+\frac{8}{4!}+\cdots\right)$$

$$=I+\frac{A}{2}\left(2I+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\cdots\right)$$

$$e^A=I+\frac{A}{2}\left(e^2-1\right):$$

Correct option is (a)

HCU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The rank of the following matrix is

$$\begin{pmatrix}2 & -2 & 0 & 6\\ 4 & 2 & 0 & 2\\ 1 & -1 & 0 & 3\\ 1 & -2 & 1 & 2\end{pmatrix}$$

(a) 2 (b) 3 (c) 1 (d) 4 **[H.C.U.-2010]**

Soln: Given matrix is $A=\begin{pmatrix}2 & -2 & 0 & 6\\ 4 & 2 & 0 & 2\\ 1 & -1 & 0 & 3\\ 1 & -2 & 1 & 2\end{pmatrix}$

$$|A|_{4\times4}=2\begin{vmatrix}2&0&2\\-1&0&3\\-2&1&2\end{vmatrix}-(-2)\begin{vmatrix}4&0&2\\1&0&3\\1&1&2\end{vmatrix}+0\begin{vmatrix}4&2&2\\1&-1&3\\1&-2&2\end{vmatrix}-6\begin{vmatrix}4&2&0\\1&-1&0\\1&-2&1\end{vmatrix}$$

$$=2(-1(6+2))+2(-1(12-2))-6(1(-4-2))$$

$$=-16-20+36=0$$

Let us take a 3×3 submatrix

$$A'=\begin{pmatrix}4&2&0\\1&-1&0\\1&-2&1\end{pmatrix};\ |A'|_{3\times3}=1(-4-2)=-6\neq0$$

Hence rank of A is 3.

Correct option is (b)

Question 2. It is given that the matrix

$$\begin{pmatrix}1&0&0\\0&\alpha&-\alpha\\0&\alpha&\alpha\end{pmatrix}$$

is orthogonal. The value of α must be equal to

(a) 1 (b) 0 (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$ **[H.C.U.-2010]**

Soln: If a matrix is orthogonal, then it's each row or column is a vector having unit magnitude.

Let us consider vector associated with second column.

$$\vec{c}_2=0\hat{i}+\alpha\hat{j}+\alpha\hat{k}$$

$$\therefore\ |\vec{c}_2|=\sqrt{d^2+\alpha^2}=1\Rightarrow\alpha=\frac{1}{\sqrt{2}}$$

Correct option is (d)

Question 3. If A and B are $n\times n$ matrices, which of the following does not equal $(A+B)^2$?

(a) $A^2+2AB+B^2$ (b) $A^2+AB+BA+B^2$

(c) $(A+B)A+(A+B)B$ (d) $(B+A)^2$ **[H.C.U.-2011]**

Soln: $(A+B)^2=(A+B)(A+B)$

$$=A^2+AB+BA+B^2$$

$= (A+B)\cdot A + (A+B)B = (B+A)^2$

Hence, option (a) is not equal to $(A+B)$

Correct option is (a)

Question 4. What should be the value of α so that the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -\alpha \\ 0 & \alpha & \alpha \end{pmatrix}$ can become orthogonal?

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\sqrt{2}$ **[H.C.U.-2011]**

Soln: If a matrix is orthogonal, then it's each row or column is a vector having unit magnitude.
Let us consider vector associated with second column.

$\vec{c}_2 = 0\hat{i} + \alpha\hat{j} + \alpha\hat{k}$

$\therefore \ |\vec{c}_2| = \sqrt{d^2 + \alpha^2} = 1 \Rightarrow \alpha = \frac{1}{\sqrt{2}}$

Correct option is (d)

Question 5. The eigenvalues of the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ are given by

(a) 1, 0, 1 (b) 0, 1, 2 (c) 1, 2, 1 (d) 1, 1, 1 **[H.C.U.-2011]**

Soln: $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Sum of eigenvalues = sum of principle diagonal element.
= 3.

Product of eigenvalues = det (A)

$= 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

$= 1 - 1 = 0$

Therefore, option (b) satisfies these both condition.

Correct option is (b)

Question 6. Which one of the following statements is true for the matrix given below?

$A = \begin{pmatrix} 0 & 1 \\ -1 & i \end{pmatrix}$ **[H.C.U.-2012]**

(a) A is symmetric (b) A is hermitian (c) A is anti-symmetric (d) A is skew-Hermitian

Soln: $A = \begin{pmatrix} 0 & 1 \\ -1 & i \end{pmatrix}$

$\because\ (A^T)^C = -A$, Hence A is Skew Hermitian.

Correct option is (d)

Question 7. Which of the following cannot possibly be the eigenvalues of a real 3×3 matrix?

(a) $1, 2, -1$ (b) $1, 1+i, 1-i$ (c) $1, 0, -1$ (d) $-1, 1+i, -1+i$ **[H.C.U.-2012]**

Soln: For a real matrix, sum of eigenvalues must be real. Hence option (d) cannot be eigenvalues.

Correct option is (d)

Question 8. The system of linear equations

$$\begin{pmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 7 \end{pmatrix}$$

(a) has infinite number of solutions (b) has only trivial solution

(c) has a unique solution (d) is inconsistent and has no solution **[H.C.U.-2013]**

Soln: $D = \begin{pmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 1 \end{pmatrix} \Rightarrow |D| = \begin{vmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 1 \end{vmatrix}$

$= 4(3-6) - 9(2-2) + 3(12-6)$

$= -12 - 0 + 18 = 6 \neq 0$

Therefore, given sytem of linear equation will have unique solution.

Correct option is (c)

Question 9. Which of the following are eigenvalues of the matrix $\begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$

(a) 1, 0 (b) $i, -i$ (c) $1+i, 1-i$ (d) 1, 3 **[H.C.U.-2014]**

Soln: Sum of eigenvalues = sum of principle diagonal element.

Correct option is (d)

Question 10. What value of α will make the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -\alpha \\ 0 & \alpha & \alpha \end{pmatrix}$ orthogonal

(a) $\sqrt{2}$ (b) 1 (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2}$ **[H.C.U.-2015]**

Soln: If a matrix is orthogonal, then it's each row or column is a vector having unit magnitude.

Let us consider vector associated with second column.

$\vec{c}_2 = 0\hat{i} + \alpha\hat{j} + \alpha\hat{k}$

$\therefore\ |\vec{c}_2| = \sqrt{d^2 + \alpha^2} = 1 \Rightarrow \alpha = \frac{1}{\sqrt{2}}$

Correct option is (d)

Question 11. If $abc = P$ and $A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$, where A is an orthogonal matrix, then the value of $a + b + c$ is

(a) 2 (b) P (c) 2P (d) ± 1 **[H.C.U.-2015]**

Soln: $A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$ is an orthogonal matrix.

Hence each row or column is a unit vector and every rows and columns are orthogonal to each.

$\therefore \ \vec{R}_1 = a\hat{i} + c\hat{j} + b\hat{k}$ and $\vec{R}_2 = b\hat{i} + a\hat{j} + c\hat{k}$

$\therefore \ \sqrt{a^2 + b^2 + c^2} = 1$ and $\vec{R}_1 \cdot \vec{R}_2 = 0$ or $a^2 + b^2 + c^2 = 1$ ... (1)

$ab + ac + bc = 0$

$ab + bc + ca = 0$ ... (2)

Now, $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 1 \neq 0$

$\therefore \ a + b + c = \pm 1.$

Correct option is (d)

Question 12. The eigenvalues of the matrix, $A = \begin{pmatrix} 5 & -2 \\ 0 & 0 \end{pmatrix}$ are

(a) 0, 0 (b) 5, 0 (c) 0, 6 (d) 5, 6 **[H.C.U.-2016]**

Soln: Sum of eigenvalues = sum of principle diagonal element.

Correct option is (b)

Question 13. The determinant of the matrix $A = \begin{pmatrix} e^{i\pi} & -1 \\ 1 & 1 \end{pmatrix}$ is

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(a) 1 (b) π (c) $i\pi$ (d) 0 **[H.C.U.-2016]**

Soln: $A = \begin{pmatrix} e^{i\pi} & -1 \\ 1 & 1 \end{pmatrix}$

$A = \begin{vmatrix} e^{i\pi} & -1 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} = 0$

Correct option is (d)

Question 14. The eigenvalues of the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are **[H.C.U.-2017]**

(a) $e^{\pm i0}$ (b) $e^{\pm 2i0}$ (c) $e^{\pm 3i0}$ (d) $e^{\pm i0/2}$

Soln: If eigenvalues are $e^{\pm i0}$, i.e. $\lambda_1 = e^{i0}$ and $\lambda_2 = e^{-i0}$

$\therefore \ \lambda_1 + \lambda_2 = e^{i\theta} + e^{-i0} = 2\cos\theta =$ sum of principle diagonal element.

Correct option is (a)

Question 15. The eigenvectors of the matrix $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$ are **[H.C.U.-2017]**

(a) $a_1=\begin{bmatrix}1\\0\end{bmatrix}$, $a_2=\begin{bmatrix}-1\\0\end{bmatrix}$ (b) $a_1=\begin{bmatrix}1\\1\end{bmatrix}$, $a_2=\begin{bmatrix}1\\-1\end{bmatrix}$

(c) $a_1=\begin{bmatrix}2\\0\end{bmatrix}$, $a_2=\begin{bmatrix}0\\2\end{bmatrix}$ (d) $a_1=\begin{bmatrix}1\\0\end{bmatrix}$, $a_2=\begin{bmatrix}0\\1\end{bmatrix}$

Soln: $A=\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$. Therefore, characteristic equation, $|A-\lambda I|=0 \Rightarrow \begin{vmatrix}2-\lambda & -1\\ -1 & 2-\lambda\end{vmatrix}=0 \Rightarrow (2-\lambda)^2-1=0$

$4+\lambda^2-4\lambda-1=0 \Rightarrow \lambda^2-4\lambda+3=0$

$\therefore\ \lambda=3,1$

$\therefore$ eigenvector corresponding to $\lambda=3$

$\Rightarrow\begin{pmatrix}2-3 & -1\\ -1 & 2-3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0 \Rightarrow -x-y=0\ ;\ x=-y$

Hence, if $x=1,\ y=-1$. Therefore, eigenvector $\begin{pmatrix}1\\-1\end{pmatrix}$

Eigenvector corresponding to $\lambda=1$.

$\begin{pmatrix}2-1 & -1\\ -1 & 2-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow x-y=0\ ;\ x=y$

Hence, if $x=1,\ y=1$. Therefore, eigenvector $\begin{pmatrix}1\\1\end{pmatrix}$

Correct option is (b)

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BHU PREVIOUS YEAR QUESTIONS & SOLUTINS

Question 1. If a matrix is $A=\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}$, then A^{-1} will be **[B.H.U-2010]**

(a) $-\frac{1}{5}\begin{pmatrix}1 & -3\\ -2 & 1\end{pmatrix}$ (b) $\frac{1}{5}\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}$ (c) $\frac{1}{4}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ (d) $-\frac{1}{3}\begin{pmatrix}-1 & 3\\ 2 & -1\end{pmatrix}$

Soln: $A=\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}$; It's characteristic equation, $|A-\lambda I|=0 \Rightarrow \begin{vmatrix}1-\lambda & 3\\ 2 & 1-\lambda\end{vmatrix}=0$

$\Rightarrow 1+\lambda^2-2\lambda-6=0 \Rightarrow \lambda^2-2\lambda-5=0$

Hence, from Caley Hamilton theorem, $A^2-2A-5=0$,

$A^2A^{-1}-2AA^{-1}-5A^{-1}=0\ ;\ A-2I=5A^{-1}$

$$A^{-1}=\frac{1}{5}\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}-\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-1 & 3\\ 2 & -1\end{pmatrix}.$$

Correct option is (a)

Question 2. If a matrix is $A=\begin{pmatrix}3 & 1\\ 2 & 2\end{pmatrix}$, its eigenvalues are **[B.H.U-2010]**

(a) 1 and 2 (b) 2 and 4 (c) 1 and 3 (d) 1 and 4

Soln: Sum of eigenvalues = trace of $A = 5$. Therefore, only option (d) is satisfying this one.
Correct option is (d)

Question 3. The matrix **[B.H.U-2011]**

$$A=\begin{bmatrix}\cos\theta & i\sin\theta\\ i\sin\theta & \cos\theta\end{bmatrix} \text{ is}$$

(a) Hermitian (b) unitary (c) orthogonal (d) symmetric

Soln: $A=\begin{bmatrix}\cos\theta & i\sin\theta\\ i\sin\theta & \cos\theta\end{bmatrix}$

Vector corresponding to 1st and 2nd rows are $\vec{R}_1=\cos\theta\,\hat{i}+(i\sin\theta)\,\hat{j}$ and $\vec{R}_2=(i\sin\theta)\,\hat{i}+\cos\theta\,\hat{j}$.

Clearly, $|\vec{R}_1|=|\vec{R}_1|=1$ and $\vec{R}_1\cdot\vec{R}_2=0$.

Hence, unitary.
Correct option is (c)

Question 4. Indicate the statement which is not true **[B.H.U-2011]**

(a) the diagonalizing matrix of a unitary matrix is Hermitian
(b) the diagonalizing matrix of a real symmetric matrix is orthogonal
(c) the non-zero elements of a diagonal matrix are its eigenvalues
(d) the diagonalizing matrix of a Hermitian matrix is unitary

Soln: **Correct option is (a)**

Question 5. A matrix $C=\begin{pmatrix}5 & -1\\ 1 & 3\end{pmatrix}$ has **[B.H.U-2012]**

(a) no independent eigenvectors (b) 1 independent eigenvector
(c) 2 independent eigenvectors (d) 3 independent eigenvector

Soln: $C=\begin{pmatrix}5 & -1\\ 1 & 3\end{pmatrix}$; Eigenvalue equation is $\begin{vmatrix}5-\lambda & -1\\ 1 & 3-\lambda\end{vmatrix}=0\Rightarrow(5-\lambda)(3-\lambda)+1=0$

$15-8\lambda+\lambda^2+1=0\Rightarrow\lambda^2-8\lambda+16=0\Rightarrow\lambda=4,4.$

Since, eigenvalues are same, hence there would be only one independent eigenvectors.
Correct option is (b)

Question 6. The eigenvalues of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ are [B.H.U-2012]

(a) 1, 0 (b) 1, 1 (c) 1, 2 (d) 0, 2

Soln: Trace A = sum of eigenvalues.
Only option (b) is satisfying this.
Correct option is (b)

Question 7. Eigenvalues of the matrix $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are [B.H.U-2013]

(a) $1, 0, -1$ (b) $1, 1, 0$ (c) $2, 0, -2$ (d) $2, 0, 2$

Soln: Trace A = sum of eigenvalues. No options are satisfying this.

Question 8. Which of the following vectors is not an eigenvector of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$? [B.H.U-2013]

(a) $\vec{r} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ (b) $\vec{r} = (0, 0, 1)$

(c) $\vec{r} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ (d) $\vec{r} = (1, 0, -1)$

Soln: $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; It's eigenvalue $\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$

$\Rightarrow -\lambda\left(\lambda^2\right) - 1\left(-\lambda\right) = 0 \Rightarrow -\lambda^3 + \lambda = 0 \Rightarrow \lambda\left(1 - \lambda^2\right) = 0 \Rightarrow \lambda = 0, 1, -1.$

Eigenvector corresponding to $\lambda = 0$

$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \boxed{y = 0}\ \boxed{x = 0}$, z can be anything.

Therefore, option (b) is related to this.

Eigenvector corresponding to $\lambda = 1$.

$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$-x + y = 0 \Rightarrow \boxed{x = y}$; $x - y = 0$; $-z = 0 \Rightarrow \boxed{z = 0}$

Therefore, (a) is related to this.

Eigenvector corresponding to $\lambda = -1$.

$\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=0$; $x+y=0$ $\boxed{x=-y}$ $\boxed{z=0}$

Therefore, option (c) is related to this.

Correct option is (d)

Question 9. For any square matrix A, which of the following matrices is not Hermitian ?**[B.H.U-2013]**

(a) $A+A^{+}$ (b) AA^{+} (c) $A^{+}A$ (d) $A^{+}-A$

Soln: Check option (d) ; $\left(A^{+}-A\right)^{+}=\left(A^{+}\right)^{+}-A^{+}=A-A^{+}=-\left(A^{+}-A\right)$

Correct option is (d)

Question 10. A 2×2 matrix has determinant 1 and trace 2. Its eigen values are :

(a) ± 1 (b) 0, 1 (c) 1, 1 (d) 0, 2 **[B.H.U-2014]**

Soln: Let eigenvalues are λ_1 and λ_2, then $\text{tr}(A)=\lambda_1+\lambda_2 \Rightarrow \lambda_1+\lambda_2=2$ ... (1)

and $\det(A)=\lambda_1\lambda_2 \Rightarrow \lambda_1\lambda_2=1$ ... (2)

Correct option is (c)

Question 11. Eigenvalues of $\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$ are **[B.H.U-2015]**

(a) $-1, 0, 2$ (b) 0, 1, 2 (c) $-1, 1, 2$ (d) $-3, 1, 5$

Soln: Sum of eigenvalues = trace A and product of eigenvalues = det (A)

Therefore, option (b) satisfies these two conditions.

Correct option is (b)

Question 12. If A is 3×3 matrix with Tr(A) = 3, det A = 0 and one of the eigenvalue 1, the other two eigenvalues are

(a) 2 and 0 (b) 1 and 1 (c) 0 and 1 (d) 3 and -1 **[B.H.U-2016]**

Soln: Sum of eigenvalues = trace A

Product of eigenvalues = det A

Therefore, option (a) satisfies these two conditions.

Correct option is (a)

Question 13. If A is a skew-symmetric matrix and R is its rank, then

(a) $R=1$ (b) $R\le 1$ (c) $R>1$ (d) $R\ge 1$ **[B.H.U-2016]**

Soln: Rank of a matrix is always greater than one.

Correct option is (c)

Question 15. If I and O are (2×2) identity and null matrices in $I=pA+qA^2=0$ then, p and q values for the matrix. $A=\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}$ are one of the following: **[B.H.U-2016]**

(a) $p=\frac{2}{5}, q=\frac{1}{5}$ (b) $p=-\frac{2}{5}, q=\frac{1}{5}$ (c) $p=\frac{2}{5}, q=-\frac{1}{5}$ (d) $p=-\frac{2}{5}, q=-\frac{1}{5}$

Soln: $A=\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}$; $I+pA+qA^2=0$

$$I+p\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}+q\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}=0$$

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+p\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}+q\begin{pmatrix}-3 & 4\\ -4 & -3\end{pmatrix}=0$$

$$\begin{pmatrix}p-3q+1 & 2p+4q\\ -2p-4q & p-3q+1\end{pmatrix}=0$$

$p-3q+1=0$ ... (1)

$2p+4p=0$ ... (2)

From Eq. (1) and Eq. (2); $-2q-3q+1=0$

$\Rightarrow q=\frac{1}{5} \Rightarrow p=-\frac{2}{5}$.

Correct option is (b)

Question 16. The eigenvectors of the matrix $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ are as follows: **[B.H.U-2016]**

(a) $\begin{pmatrix}1\\ i\end{pmatrix}$ and $\begin{pmatrix}1\\ -i\end{pmatrix}$ (b) $\begin{pmatrix}1\\ i\end{pmatrix}$ and $\begin{pmatrix}-1\\ i\end{pmatrix}$ (c) $\begin{pmatrix}i\\ 1\end{pmatrix}$ and $\begin{pmatrix}-i\\ 1\end{pmatrix}$ (d) $\begin{pmatrix}i\\ 1\end{pmatrix}$ and $\begin{pmatrix}i\\ -1\end{pmatrix}$

Soln: $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$; Eigenvalue equation is $|A-\lambda I|=0$

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$$\begin{vmatrix}\cos\theta-\lambda & -\sin\theta\\ \sin\theta & \cos\theta-\lambda\end{vmatrix}=0$$

$\cos^2\theta+\lambda^2-2\cos\theta\lambda+\sin^2\theta=0$

$\lambda^2-2\cos\theta\lambda+1=0$

$$\lambda=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}=\frac{2\cos\theta\pm 2i\sin\theta}{2}$$

$\lambda=\cos\theta\pm i\sin\theta \Rightarrow \lambda=e^{i\theta}, e^{-i\theta}$

Eigenvector corresponding to $\lambda=e^{i\theta}$.

$$(A-\lambda I)X=0 \Rightarrow \begin{pmatrix}\cos\theta-e^{i\theta} & \sin\theta\\ -\sin\theta & \cos\theta-e^{i\theta}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=0 \Rightarrow \begin{pmatrix}-i\sin\theta & \sin\theta\\ -\sin\theta & -i\sin\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=0$$

$-ix+y=0$ $\boxed{x=-iy}$, if $y=1, x=-i$

Therefore, $X_{\lambda=e^{i\theta}} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

Eigenvector corresponding to $\lambda = e^{i\theta}$; $(A-\lambda I)X = 0$

$$\begin{pmatrix} \cos\theta - e^{-i\theta} & \sin\theta \\ -\sin\theta & \cos\theta - e^{-i\theta} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} i\sin\theta & \sin\theta \\ -\sin\theta & i\sin\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$ix + y = 0$ $\boxed{x = iy}$; if $y = 1$, $x = i$

$$X_{\lambda=e^{-i\theta}} = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Hence, two eigenvectors are $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

Correct option is (c)

Question 17. Transpose conjugate of two matrices A and B i.e., $(AB)^+$ equals **[B.H.U-2017]**

(a) A^+B^+ (b) B^+A^+ (c) $B'A'$ (d) AB

Soln: $(AB)^+ = B^+A^+$

Correct option is (b)

Question 18. The product of a singular matrix with its adjoint gives **[B.H.U-2017]**

(a) a unitary matrix (b) a null matrix (c) a diagonal matrix (d) None of these

Soln: We have $A(\text{adj}\,A) = |A|\,I$. Since, A is a singular matrix, $|A| = 0$

Therefore, $A(\text{adj}\,A) = 0$ null matrix.

Correct option is (b)

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Chapter 3

DIFFERENTIAL EQUATIONS

3.1 Basic Review of Differential Equation

Definition:

An equation which involves differential co-efficient is called differential equation.

Example: $\frac{dy}{dx}=\frac{1-x^2}{1+y^2}$; $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+6y=0$

All differential equations can be classified into two types:

(i) Ordinary differential equation: This consists of derivatives w.r.t. a single independent variable.

(ii) Partial differential equation: This consists of partial derivatives w.r.t. more than one independent variable.

Order and Degree of a differential equation:

The order of the highest order derivative present in the given equation is known as order of the differential equation and the power of the highest order derivative term present in the given equation is known as degree of differential equation.

Example: $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=E\sin\omega t$ $\Rightarrow$ order 2; degree 1

$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^2y}{dx^2}\right)^2$ $\Rightarrow$ order 2, degree 2

Linear Differential Equation:

A differential equation is said to be linear if (i) every dependent variable and its derivative occurs in first degree only and (ii) no products of dependent variables and its derivatives occur.

Example: $\frac{dy}{dx}=(x+\sin x)$, $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+7y=0$

Family of Curves:

An n-parameter family of curves is a set of relations of the form

$$\{(x,y): f(x,y,c_1,c_2.....c_n)\}=0$$

where 'f' is a real valued function of $x, y, c_1, c_2.....c_n$ and each c_i ranges over an interval of real values.

Example: The set of concentric circles is defined by $x^2 + y^2 = c$ is a one parameter family if c takes all positive real values.

There will be n arbitary constants in n-parameter family of curves. Then, we can obtain n^{th} order differential equation whose solution is the given family.

Example 1: The differential equation of all circles passing through the origin and having their centres on the y-axis

(a) $\left(x^2 + y^2\right)dy/dx = 2xy$ (b) $\left(x^2 - y^2\right)dy/dx = 2xy$

(c) $dy/dx = 2xy\left(x^2 + y^2\right)$ (d) $dy/dx = 2xy\left(x^2 - y^2\right)$

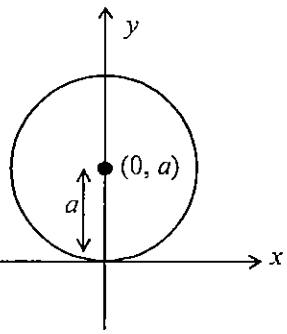
Soln: Equation of the circle passing through the origin and having their centers on the y-axis will be of the form

$$x^2 + (y-a)^2 = a^2$$

$$\Rightarrow 2x + 2(y-a)\frac{dy}{dx} = 0 \Rightarrow (y-a) = -\frac{x}{(dy/dx)} \Rightarrow a = y + \frac{x}{(dy/dx)}$$

Therefore, $x^2 + \frac{x^2}{(dy/dx)^2} = \left(y + \frac{x}{(dy/dx)}\right)^2$

$$\Rightarrow x^2 = y^2 + \frac{2xy}{(dy/dx)} \Rightarrow \left(x^2 - y^2\right)\frac{dy}{dx} = 2xy$$



Correct option is (b)

Example 2: The function $y = \ln\left[\sin(x+a)\right] + b$ is the solution of the following differential equation:

(a) $y'' = -\left[1 + \left(y'\right)^2\right]$ (b) $y'' = y^2 - \left(y'\right)^2$ (c) $y'' = \left[1 + \left(y'\right)^2\right]$ (d) $y'' = y' + y^2$

Soln: $y = \ln\left[\sin(x+a)\right] + b \Rightarrow \frac{dy}{dx} = \cot(x+a) \Rightarrow \frac{d^2y}{dx^2} = -\operatorname{cosec}^2(x+a) = -\left[1 + \cot^2(x+a)\right]$

Therefore, $\frac{d^2y}{dx^2} = -\left[1 + \left(\frac{dy}{dx}\right)^2\right]$

Correct option is (a)

3.2 Differential Equations of the First Order and First Degree:

CASE I: Separation of Variables

Every Differential equation can be written in the form,

$$f(y)dy = \phi(x)dx \Rightarrow \int f(y)dy = \int \phi(x)dx + C$$

Example 3: Solve $e^{dy/dx} = (x+1)$; given $y = 3$ at $x = 0$

Soln: Taking log of both sides we get, $\frac{dy}{dx} = \ln(x+1) \Rightarrow dy = \ln(x+1)\,dx$

on integration, $\int dy = \int 1.\ln(x+1)dx$

$$\Rightarrow \quad y = x\ln(x+1) - \int \frac{x}{(x+1)}dx + C \quad \Rightarrow \quad y = x\ln(x+1) - \int \frac{(x+1)-1}{(x+1)}dx + C$$

$$\Rightarrow \quad y = x\ln(x+1) - \int \frac{(x+1)}{(x+1)}dx + \int \frac{1}{(x+1)}dx + C \quad = x\ln(x+1) - x + \ln(x+1) + C$$

$$\Rightarrow \quad y = (x+1)\ln(x+1) - x + C$$

Given: at $x = 0 \quad y = 3 \Rightarrow \quad C = 3$

Therefore, $y = (x+1)\ln(x+1) - x + 3$

Example 4: Solve $\frac{dy}{dx} = e^{y}\left(e^{x} + x^{2}e^{x^3}\right)$

Soln: $\frac{dy}{dx} = e^{y}\left(e^{x} + x^{2}e^{x^3}\right) \Rightarrow \int e^{-y}dy = \int\left(e^{x} + x^{2}e^{x^3}\right)dx \Rightarrow -e^{-y} = e^{x} + (1/3)e^{x^3} + C$

Example 5: Solve $(x+y)(dx-dy) = dx+dy$

Soln: Re-writting the given equation, we get $(x+y-1)\,dx = (x+y+1)\,dy \Rightarrow \quad \frac{dy}{dx} = \frac{x+y-1}{x+y+1}$

Let $x+y=v \Rightarrow \frac{dy}{dx} = (dv/dx) - 1$

Therfore the given equation becomes,

$$\frac{dv}{dx} - 1 = \frac{v-1}{v+1} \Rightarrow \frac{dv}{dx} = \frac{2v}{v+1} \Rightarrow \int 2dx = \int\left(1 + \frac{1}{v}\right)dv$$

$$\Rightarrow 2x + c = v + \ln v \quad \Rightarrow -y + c = \ln(x+y)$$

Example 6: Solve $\frac{dy}{dx} = \frac{4x+6y+5}{3y+2x+4}$

Soln: The given equation may be re-written as $\frac{dy}{dx} = \frac{2(2x+3y)+5}{(2x+3y)+4}$

Taking $2x+3y=v \Rightarrow 2+3\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

Therefore, the given equation becomes,

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{2v+5}{v+4} \Rightarrow \frac{dv}{dx} = \frac{3(2v+5)}{v+4} + 2 = \frac{8v+23}{v+4}$$

$$\Rightarrow \frac{dx}{dv}=\frac{v+4}{8v+23}=\frac{\frac{1}{8}(8v+23)+\left(4-\frac{23}{8}\right)}{8v+23}=\left[\frac{1}{8}+\frac{9}{8(8v+23)}\right] \Rightarrow \int dx=\int\left[\frac{1}{8}+\frac{9}{8(8v+23)}\right]dv$$

$$\Rightarrow x+c=\left(\frac{v}{8}\right)+\left(\frac{9}{64}\right)\log(16x+24y+23) \quad \Rightarrow 3y-6x+\left(\frac{9}{8}\right)\log(16x+24y+23)=8c$$

$$\Rightarrow y-2x+\left(\frac{3}{8}\right)\log(16x+24y+23)=c' \text{ where } c'(=8c/3) \text{ is an arbitrary constant.}$$

Example 7: Solution of the differential equation $\frac{dy}{dx}=\sin(x+y)+\cos(x+y)$ is

(a) $x+c=\log\left[1+\tan\left(\frac{x+y}{2}\right)\right]$ (b) $x+c=\log\left[1-\tan\left(\frac{x+y}{2}\right)\right]$

(c) $x+c=\log\left[1+\cot\left(\frac{x+y}{2}\right)\right]$ (d) $x+c=\log\left[1-\cot\left(\frac{x+y}{2}\right)\right]$

Soln: Put $x+y=z \Rightarrow 1+\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{dy}{dx}=\frac{dz}{dx}-1$

Hence, given differential equation can be written as,

$$\frac{dz}{dx}-1=\sin z+\cos z \quad \Rightarrow \int\frac{dz}{1+\sin z+\cos z}=\int dx$$

$$\Rightarrow \int\frac{dz}{1+\frac{2\tan\frac{z}{2}}{1+\tan^2\frac{z}{2}}+\frac{1-\tan^2\frac{z}{2}}{1+\tan^2\frac{z}{2}}}=x+c$$

$$\Rightarrow \int\frac{\sec^2\frac{z}{2}dz}{1+\tan^2\frac{z}{2}+2\tan\frac{z}{2}+1-\tan^2\frac{z}{2}}=x+c \Rightarrow \int\frac{\sec^2\frac{z}{2}dz}{1+2\tan\frac{z}{2}}=x+c$$

Put $1+2\tan\frac{z}{2}=t \Rightarrow 2\times\sec^2\frac{z}{2}\frac{dz}{2}=dt$

$\therefore \quad \int\frac{dt}{t}=x+c \Rightarrow \log(t)=x+c \Rightarrow x+c=\log\left[1+2\tan\left(\frac{x+y}{2}\right)\right]$.

Correct option is (a).

Example 8: Which of the following statement(s) about the solution $y(x)$ of the differential equation $\frac{1}{x^2-3}\frac{dy}{dz}=1$ is correct (with $y(0)=0$)

(a) $y(x)$ is given by $\frac{x^3}{3}-3x$

(b) $y(x)$ is given by $\frac{x^2}{2}-2x$

(c) $y(x)$ has extremum points at $\sqrt{3}$ and $-\sqrt{3}$

(d) $y(x)$ is increasing in the interval $\left(-\infty,-\sqrt{3}\right)\cup\left(\sqrt{3},\infty\right)$

Soln: We have $\int dy=\int(x^2-3)\,dx \Rightarrow y=\frac{x^3}{3}-3x+c$

Since, $y(0)=0$, Put $x=0,\ y=0 \Rightarrow c=0$. Therefore, $y(x)=\frac{x^3}{3}-3x$

Now for extremum points, $y'(x)=0 \Rightarrow x^2-3=0,\ x=\pm\sqrt{3}$

Now using sign scheme for $y'(x)$,

We have $\underset{-\sqrt{3}}{+}\quad\underset{\sqrt{3}}{-}\quad+$

So, $y(x)$ is increasing in the interval $\left(-\infty,-\sqrt{3}\right)\cup\left(\sqrt{3},0\right)$

Correct options are (a), (c) and (d)

Example 9: The solution of the differential equation $\frac{dx}{dt}=x^2$ with the initial condition x(0) =1 will blow up as t tend to

(a) 1 (b) 2 (c) 1/2 (d) ∞

Soln: Given: $\frac{dx}{dt}=x^2 \Rightarrow \int\frac{dx}{x^2}=\int dt \Rightarrow -\frac{1}{x}=t+C$

Applying $x(0)=1$, we get $C=1 \Rightarrow -\frac{1}{x}=t-1 \Rightarrow x=\frac{1}{1-t}$

Therefore, the solution of the given differntial equation will blow up as t tends to 1.

Correct answer is (a)

Example 10: The solutions to the differential equation $\frac{dy}{dx}=-\frac{x}{y+1}$ are a family of

(a) Circles with different radii (b) Circles with different centres.

(c) Straight lines with different slopes (d) Straight lines with different intercepts on the y-axis

Soln: $\frac{dy}{dx}=-\frac{x}{y+1} \Rightarrow \int(y+1)\,dy=-\int x\,dx$

$\Rightarrow \frac{y^2}{2}+y=-\frac{x^2}{2}+c$

$\Rightarrow x^2+y^2+2y=2c$

$\Rightarrow x^2+(y+1)^2=2c+1$

This represents a family of circles having $(0,-1)$ and radius of $\sqrt{2c+1}$ units.

Correct option is (a)

Example 11: Consider the linear differential equation $\frac{dy}{dx} = xy$. If $y=2$ at $x=0$, then the value of y at $x=$ 2 is

(a) e^{-2} (b) $2e^{-2}$ (c) e^{2} (d) $2e^{2}$

Soln: We have, $\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = xdx \Rightarrow \ln y = \frac{x^2}{2} + c$

Given, $x = 0, y = 2, \therefore c = \ln 2$

$\Rightarrow \quad \ln y = \frac{4}{2} + \ln 2 \quad \Rightarrow \quad \ln\left(\frac{y}{2}\right) = 2$

$\Rightarrow \quad y = 2e^2$

Correct option is (d)

CASE II: Homogeneous Differential equation:

A differential equation $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ is said to homogeneous if $f(x, y)$ and $\phi(x, y)$ are of same degree.

In such equations, put $y = vx$ & $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Example 12: Solve $xdy - ydx = \sqrt{\left(x^2+y^2\right)}dx$

Soln: $\frac{dy}{dx} = \frac{y+\sqrt{\left(x^2+y^2\right)}}{x} = \frac{y}{x} + \sqrt{\left\{1+\left(\frac{y}{x}\right)^2\right\}}$

Take, $\frac{y}{x} = v, i.e.\ y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = v + \sqrt{\left(1+v^2\right)} \Rightarrow \int\frac{dx}{x} = \int\frac{dv}{\sqrt{\left(1+v^2\right)}} \quad \Rightarrow \log x + \log c = \log\left[v + \sqrt{\left(v^2+1\right)}\right]$

$\Rightarrow xc = v + \sqrt{\left(v^2+1\right)} \quad \Rightarrow x^2c = y + \sqrt{\left(y^2+x^2\right)}$

Example 13: Solve $x\cos\left(\frac{y}{x}\right)\left(ydx + xdy\right) = y\sin\left(\frac{y}{x}\right)\left(xdy - ydx\right)$

Soln: $\frac{dy}{dx} = \frac{\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}x} \quad \Rightarrow \frac{dy}{dx} = \frac{\left[\cos\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)\right]\left(\frac{y}{x}\right)}{\left[\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)\right]}$

Take, $\frac{y}{x} = v\ i.e.\ y = vx \quad \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v(\cos v+v\sin v)}{v\sin v-\cos v} \Rightarrow x\frac{dv}{dx}=\frac{v(\cos v+v\sin v)}{v\sin v-\cos v}-v=\frac{2v\cos v}{v\sin v-\cos v}$$

$$\Rightarrow \int 2\frac{dx}{x}=\int\frac{v\sin v-\cos v}{v\cos v}dv=\int\left[\frac{\sin v}{\cos v}-\frac{1}{v}\right]dv \Rightarrow 2\log x=-\log\cos v-\log v+\log c$$

$$\Rightarrow \log x^2=\log(c/v\cos v) \Rightarrow xy\cos\frac{y}{x}=c$$

Example 14: Solve $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

Soln: Take $x=X+h, y=Y+k$, so that $\frac{dy}{dx}=\frac{dY}{dX}$ ... (1)

Therefore, given equation becomes $\frac{dy}{dx}=\frac{X+2Y+(h+2k-3)}{2X+Y+3+(2h+k-3)}$ ... (2)

Choose h, k so that $h+2k-3=0$ and $2h+k-3=0$ ... (3)

Solving (3) we get h = 1, k = 1 so that from (1), we have $X=x-1, Y=y-1$

Using (3) in (2), we get $\frac{dY}{dX}=\frac{X+2Y}{2X+Y}=\frac{1+(2Y/X)}{2+(Y/X)}$

Take $\frac{Y}{X}=v, i.e. Y=vX$. $\Rightarrow \frac{dY}{dX}=v+X\frac{dv}{dX}$

$$\Rightarrow v+X\frac{dv}{dX}=\frac{1+2v}{2+v} \Rightarrow X\frac{dv}{dX}=\frac{1+2v}{2+v}-v=\frac{1-v^2}{2+v}$$

$$\Rightarrow \int\frac{dX}{X}=\int\frac{(2+v)dv}{(1-v)(1+v)}=\int\left[\frac{1}{2}\left(\frac{1}{1+v}\right)+\frac{3}{2}\left(\frac{1}{1-v}\right)\right]dv$$ [Resolving into partial fractions]

$$\Rightarrow \log X+\log c=\left(\frac{1}{2}\right)\left[\log(1+v)-3\log(1-v)\right] \Rightarrow 2\log(cX)=\log\frac{1+v}{(1-v)^3} \Rightarrow X^2c^2=\frac{1+v}{(1-v)^3}$$

$$\Rightarrow X^2c^2\left(1-\frac{Y}{X}\right)^3=1+\frac{Y}{X} \Rightarrow c^2(X-Y)^3=X+Y \Rightarrow c^2\{x-y\}^2=x+y-2$$

Example 15: If $\frac{dy}{dx}=f\left(\frac{y}{x}\right)$, then the value of the integral $\int\frac{dv}{f(v)-v}$ is equal to (where $v=\frac{y}{x}$).

(a) $\log x+c$ (b) e^x+c (c) $\frac{y}{x}+c$ (d) None of these

(Here c is an integration constant)

Soln: Since, $v=\frac{y}{x}$ or $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

Now, $\frac{dy}{dx}=f\left(\frac{y}{x}\right) \Rightarrow v+x\frac{dv}{dx}=f(v) \Rightarrow \int\frac{dv}{f(v)-v}=\int\frac{dx}{x}=\log x+c$

Correct option is (a)

CASE III: Linear differential equations of first order:

$$\frac{dy}{dx}+P(x)\,y=Q(x)$$

where P and Q are the functions of x only or constant.

For the solution of these equations we have to calculate integral factor $(I.F)=e^{\int P(x)dx}$ and the solution of differential equation will be of the form

$$I.F.\times y=\int Q(x)\times I.F.dx+C$$

Example 16: Solve $(x+1)\frac{dy}{dx}-y=e^x(x+1)^2$

Soln: The equation can be rewritten as $\frac{dy}{dx}-\frac{y}{x+1}=e^x(x+1)$

Compare with the standard form of first order linear differential equation we get,

$$P=-\frac{1}{x+1}\text{ and }Q=e^x(x+1)$$

Integrating factor, $I.F.=e^{\int Pdx}=e^{-\int\frac{dx}{x+1}}=e^{-\log(x+1)}=\frac{1}{x+1}$

Therefore, the solution : $y\frac{1}{x+1}=\int e^x(x+1)\frac{1}{(x+1)}dx=\int e^x dx\Rightarrow y=e^x(x+1)+C$

Example 17: Solve $(1-x^2)\left(\frac{dy}{dx}\right)+2xy=x\sqrt{(1-x^2)}$

Soln: The given equation is $\frac{dy}{dx}+\frac{2x}{1-x^2}y=\frac{x}{\sqrt{(1-x^2)}}$

I.F. $=e^{\int Pdx}=e^{\int\frac{2x}{1-x^2}dx}=e^{-\ln(1-x^2)}=\frac{1}{1-x^2}$.

So the required solution is

$$\frac{y}{1-x^2}=\int\frac{x}{\sqrt{(1-x^2)}}\times\frac{1}{1-x^2}dx=\int\frac{x}{(1-x^2)^{3/2}}dx$$

Putting $1-x^2=t$ and $-2xdx=dt$, we get

$$\Rightarrow\frac{y}{1-x^2}=-\frac{1}{2}\int t^{-3/2}dt=t^{-1/2}+c$$

The solution is $\frac{y}{1-x^2}=\frac{1}{\sqrt{(1-x^2)}}+c$

Example 18: Integrate $\left(1+x^2\right)\left(\frac{dy}{dx}\right)+2xy-4x^2=0$. Obtain equation of the curve satisfying this equation and passing through origin.

Soln: Re-writting the given equation, $\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{4x^2}{1+x^2}$

Here, I.F. $=e^{\int P dx}=e^{\int\frac{2x}{1+x^2}dx}=e^{\ln\left(1+x^2\right)}=\left(1+x^2\right)$

Hence the requied solution,

$$y\left(1+x^2\right)=\int\frac{4x^2}{1+x^2}\left(1+x^2\right)dx+c \Rightarrow y\left(1+x^2\right)=\left(\frac{4}{3}\right)x^3+c$$

Since the required curve passes through origin i.e. for $x = 0, y = 0 \Rightarrow c = 0$

Hence, the equation of the required curve is $4x^3=3y\left(1+x^2\right)$

Example 19: The particular solution of the differential equation $3xy' - y = \ln x + 1,\ x > 0$ with the condition $y(1) = -2$, is

(a) $y = 2x^{1/3} - \ln x - 4$ (b) $y = x^{1/3} + \ln x - 4$

(c) $y = 2x^{2/3} - \ln x - 4$ (d) $y = x^{2/3} + \ln x - 4$

Soln: $y'-\frac{1}{3x}y=\frac{\ln x+1}{3x}$

Here, I.F.$= e^{\int-\frac{1}{3}x dx}=x^{-1/3}$

Therefore, the required solution will be

$$y x^{-1/3}=\int\frac{\ln x+1}{3x}\times x^{-1/3}dx+c=-x^{-1/3}(\ln x+1)-3x^{-1/3}dx+c$$

$\Rightarrow y=-(\ln x+4)+c x^{1/3}$

Since, $y(1) = -2$, putting $x = 1, y = -2$, we have $c = 2$

Therefore, $y=2x^{1/3}-(\ln x+4)$.

Correct option is (a)

Example 20: Solution of the differential equation $x^2\frac{dy}{dx}+2xy=y^3$ is

(a) $\frac{1}{5x}+c x^4$ (b) $-\frac{1}{5x}+c x^{-4}$ (c) $5x + c x^4$ (d) $5x - c x^{-4}$

Soln: $x^2\frac{dy}{dx}+2xy=y^3 \Rightarrow \frac{1}{y^3}\frac{dy}{dx}+\frac{2}{x}\times\frac{1}{y^2}=\frac{1}{x^2}$

Let $\frac{1}{y^2}=z \Rightarrow -\frac{2}{y^3}\frac{dy}{dx}=\frac{dz}{dx}$

Therefore, differential equation now becomes,

$$-\frac{1}{2}\frac{dz}{dx}+\frac{2}{x}z=\frac{1}{x^2} \Rightarrow \frac{dz}{dx}-\frac{4z}{x}=-\frac{2}{x^2}$$

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Hence, I.F.$= e^{\int -\frac{4}{x}dx} = x^{-4}$

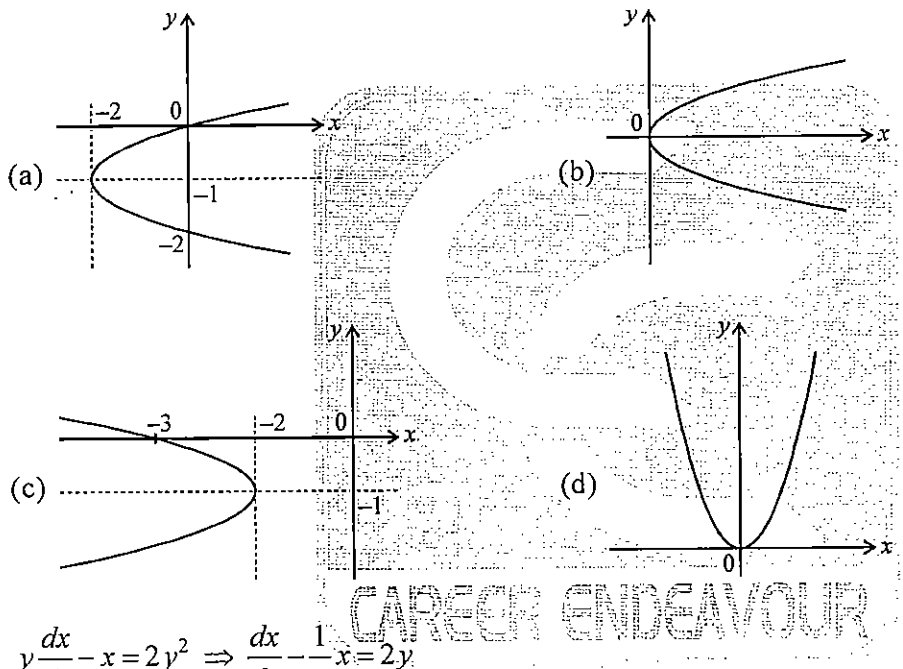
Therefore, the required solution will be

$$z \times x^{-4} = \int \frac{1}{x^2} x^{-4}\, dx + c$$

$$\Rightarrow \frac{1}{y^2} x^{-4} = -\frac{1}{5} x^{-5} + c \Rightarrow \frac{1}{y^2} = -\frac{1}{5x} + c x^4.$$

Correct option is (b)

Example 21: If $y(x)$ is the solution of the differential equation $y\frac{dx}{dy} - x = 2y^2$, $y(6) = 1$, then which of the following graph best represents $y(x)$.



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Soln: $y\frac{dx}{dy} - x = 2y^2 \Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y$

Hence, I.F. $= e^{\int -\frac{1}{y}dy} = \frac{1}{y}$.

Therefore, the requires solution will be,

$$x \times \frac{1}{y} = \int 2y \times \frac{1}{y} dy + c \Rightarrow x \times \frac{1}{y} = 2y + c \Rightarrow x = 2y^2 + cy$$

Since, $y(6) = 1$, put $x = 6$, $y = 1 \Rightarrow c = 4$

Therefore, $x = 2y^2 + 4y \Rightarrow x = 2(y^2 + 2y + 1) - 2$

$(y+1)^2 = \frac{1}{2}(x+2)$, it's a parabola with vertex at $x = -2$, $y = -1$ and symmetric about the axis $y = -1$.

Also, parabola cuts the y-axis (put $x = 0$) at $y = 0, -2$.

Correct option is (a)

Example 22: Which of the following statement(s) is/are correct for the differential equation $y' - y = 1 - e^{-x}$; $y(0) = y_0$?

(a) I.F. for this differential equation is e^{-x}

(b) I.F. for this differential equation e^{x}

(c) Value of y_0 is $-1/2$, if the given differential equation has infinite limit as $x \to \infty$

(d) Value of y_0 is $-1/2$, if the given differential equation has finite limit as $x \to \infty$

Soln: We have $y' - y = 1 - e^{-x}$

Here, I.F.$= e^{\int -dx} = e^{-x}$

Therefore, the required solution will be

$$\Rightarrow\ ye^{-x} = \int (1 - e^{-x})e^{-x}\,dx + c\ = -e^{-x} + \frac{e^{-2x}}{2} + c$$

$$\because\ y(0) = y_0 \Rightarrow y_0 = -1 + \frac{1}{2} + c \quad \boxed{c = y_0 + \frac{1}{2}}$$

$$\therefore\ ye^{-x} = -e^{-x} + \frac{1}{2}e^{-2x} + \left(y_0 + \frac{1}{2}\right)$$

$$y = -1 + \frac{e^{-x}}{2} + e^{x}\left(y_0 + \frac{1}{2}\right)$$

So, as $x \to \infty$, y will be finite, if $y_0 + \frac{1}{2} = 0 \Rightarrow y_0 = -\frac{1}{2}$.

Correct options are (a) and (c)

Example 23: If $y(x)$ is the solution of the differential equation $-x\frac{dy}{dx} + y = y^2 \log x$ with $y(1) = -1$, then

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(a) $y(x)$ is defined and finite in the range $-\infty < x \le 0$

(b) $y(x)$ is defined and finite in the range $0 < x < 3$

(c) $y(x)$ is defined and finite in the range $x \ge 3$

(d) $y(x)$ blows up at $x = e$

Soln: $-x\frac{dy}{dx} + y = y^2 \log x \Rightarrow -\frac{dy}{dx} + \frac{y}{x} = y^2\frac{\log x}{x} \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x} \times \frac{1}{y} = \frac{\log}{x}$

Let $\frac{1}{y} = z \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

Therefore, $\frac{dz}{dx} + \frac{z}{x} = \frac{\log x}{x} \Rightarrow \frac{dz}{dx} + \frac{1}{x}z = \frac{\log x}{x}$

IHere, I.F. $= e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Therefore, required solution will be

$$zx = \int \frac{\log x}{x} \times x\,dx + c \Rightarrow \frac{x}{y} = \int \log x + c \Rightarrow \frac{x}{y} = (x\log x - x) + c$$

Now, $y(1) = -1$, put $x = 1$, $y = -1$ $\Rightarrow -1 = 0 - 1 + c \Rightarrow \boxed{c = 0}$

Therefore, $\frac{x}{y} = x\log x - x \Rightarrow y = \frac{1}{\log x - 1}$

Therefore, for y to be defined $\log x$ must be defined i.e., $x > 0$

Also the solution will not be defined when $\log x - 1 = 0 \Rightarrow \log x = 1 \Rightarrow \boxed{x = e = 2.73}$

So in the range $0 < x < 3$ [option (b)], there will be a point $x = 2.73$ at which y is not defined.

Correct options are (c) and (d)

Example 24: If $y(x)$ is solution for first order differential equation $xy' + (x+1)y = 6$, then value of $y(x)$ as $x \to \infty$ is _____.

Soln: $xy' + (x+1)y = 6 \Rightarrow y' + \frac{x+1}{x}y = \frac{6}{x}$

Here, $\text{I.F.} = e^{\int \frac{x+1}{x}dx} = e^{x+\log x} = xe^x$

Therefore, $yxe^x = \int \frac{6}{x} xe^x\,dx + c = 6e^x + c \Rightarrow y = \frac{6}{x} + \frac{c}{xe^x}$

So, as $x \to \infty$, $y \to 0$

Correct answer is (0)

Example 25: Solution of the differential equation $x\frac{dy}{dx} + y = x^4$, with the boundary condition that $y = 1$ at $x = 1$, is

(a) $y = 5x^4 - 4$ (b) $y = \frac{x^4}{5} + \frac{4x}{5}$ (c) $y = \frac{4x^4}{5} + \frac{1}{5x}$ (d) $y = \frac{x^4}{5} + \frac{4}{5x}$

Soln: $x\frac{dy}{dx} + y = x^4 \Rightarrow \frac{dy}{dx} + \frac{y}{x} = x^3$

$I.F. = e^{\int dx/x} = e^{\ln x} = x$

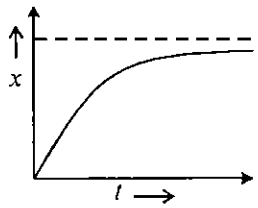
Solution will be $xy = \int x^3 x\,dx \Rightarrow xy = \frac{x^5}{5} + c$

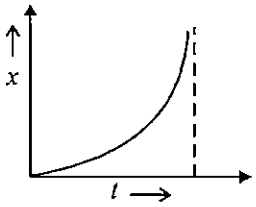
Putting the condition, at $x = 1$, $y = 1$, we get $c = \frac{4}{5}$

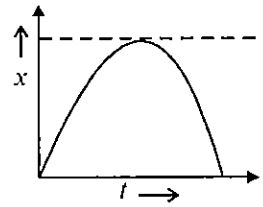
So, $xy = \frac{x^5}{5} + \frac{4}{5} \Rightarrow y = \frac{x^4}{5} + \frac{4}{5x}$

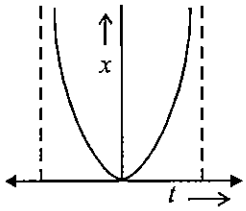
Correct option is (d)

Example 26: Which one of the following curves gives the solution of the differential equation $k_1\frac{dx}{dt}+k_2x=k_3$, where k_1, k_2 and k_3 are positive constant with initial conditions $x=0$ and $t=0$?

(a) 

(b) 

(c) 

(d) 

Soln: $k_1\frac{dx}{dt}+k_2x=k_3 \Rightarrow \frac{dx}{dt}+\frac{k_2}{k_1}x=\frac{k_3}{k_1}$

$I.F.=e^{\int\frac{k_2}{k_1}dt}=e^{(k_2/k_1)dt}$

Therefore, the solution will be

$$x\cdot e^{(k_2/k_1)t}=\int\frac{k_3}{k_1}\exp\left(\frac{k_2}{k_1}t\right)dt$$

$$\Rightarrow \quad x\cdot e^{(k_2/k_1)t}=\frac{k_3}{k_2}\exp\left(\frac{k_2}{k_1}t\right)+c$$

$$\Rightarrow \quad x=\frac{k_3}{k_2}+c\cdot\exp\left(-\frac{k_2}{k_1}t\right)$$

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Putting the condition, at $t=0$, $x=0$, we get $C=-\frac{k_3}{k_2}$

Therefore, $x=\frac{k_3}{k_2}\left[1-\exp\left(-\frac{k_2}{k_1}t\right)\right]$

Correct option is (a)

CASE IV: Exact Differential equations:

$$M(x,y)\,dx+N(x,y)\,dy=0$$

The above equation will be exact in nature if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Solution will be of the form:

$$\underset{\text{[Taking y in M as constant]}}{\int Mdx} + \underset{\text{[Take only those terms in N which do not contain x]}}{\int Ndy} = C$$

Or $$\underset{\text{[Take only those terms in N which do not contain y]}}{\int Mdx} + \underset{\text{[Taking x in N as constant]}}{\int Ndy} = C$$

Example 27: Solve $\left(x^2+2xy\right)dx+\left(x^2+y^2\right)dy=0$

Soln: Here, $M=(x^2+2xy)$ and $N=(x^2+y^2) \Rightarrow \frac{\partial M}{\partial y}=2x$ and $\frac{\partial N}{\partial x}=2x$

Hence , the given equation is exact.

The solution is: $\int (x^2+2xy)dx+\int y^2dy=C \Rightarrow \frac{x^3}{3}+x^2y+\frac{y^3}{3}=C$

Example 28: Solve $\left\{y\left(1+1/x\right)\cos y\right\}dx+\left(x+\log x-x\sin y\right)dy=0$

Soln: Here $M=y\left(1+1/x\right)+\cos y, N=x+\log x-x\sin y$ and $\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y=\frac{\partial N}{\partial x}$

Solution is $\int\left(y+y/x+\cos y\right)dx+\int 0.dy=c \Rightarrow yx+y\log x+x\cos y=c$

Example 29: If $y(x)$ is the solution of first order differential equation

$(e^{3y}-5)\,dx+3xe^{3y}\,dy=0,\ x>0$, then for $y(x)$ to be defined, x must be $[y(1)=0]$

(a) $x>4$ (b) $4\le x\le 5$ (c) $x>\frac{4}{5}$ (d) $x>5$

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Soln: $(e^{3y}-5)\,dx+3xe^{3y}\,dy=0$

Compare this with the form $M\,dx+N\,dy=0$

$M=e^{3y}-5 \Rightarrow \frac{\partial M}{\partial y}=3e^{3y}$ and $N=3xe^{3y} \Rightarrow \frac{\partial N}{\partial x}=3e^{3y}$

Since, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$; Hence, given differential equation is exact.

Hence solution is $\underset{\text{Term of } M \text{ not containing } y}{\int M\,dx}+\underset{x=\text{constant}}{\int N\,dy}=c$

$\int -5\,dx+\int 3xe^{3y}\,dy=c \Rightarrow xe^{3y}-5x=c$

Since, $y(1)=0$, put $x=1,\ y=0$

We get $c=-4$

Therefore, $xe^{3y}-5x=-4$

$$e^{3y} = \frac{-4+5x}{x} \Rightarrow y = \frac{1}{3}\ln\left(\frac{5x-4}{x}\right)$$

Thus, for $y(x)$ to be defined, $\frac{5x-4}{x} > 0 \Rightarrow x > \frac{4}{5}$.

Correct option is (c)

Example 30: Consider the differential equation $(3x^2+2xy^2)\,dx + (2x^2y)\,dy = 0$, where $y(2) = -3$. Now, choose the incorrect statement(s) :

(a) Solution $y(x)$ is given by $\sqrt{\frac{44-x^3}{x^2}}$

(b) Solution $y(x)$ is given by $-\sqrt{\frac{44-x^3}{x^2}}$

(c) Line integral of $y(x)$ over the closed curve 'c' will not be zero (where $c : x^2+y^2 = 4$)

(d) Domain of x for which $y(x)$ is real is $R - \left\{-\sqrt[3]{44}, \sqrt[3]{44}\right\} - \{0\}$

Soln: We have $(3x^2+2xy^2)\,dx + (2x^2y)\,dy = 0$

Comparing this with $M\,dx + N\,dy = 0$

$M = 3x^2+2xy^2 \Rightarrow \frac{\partial M}{\partial y} = 4xy$ and $N = 2x^2y \Rightarrow \frac{\partial N}{\partial x} = 4xy$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; Hence, it is an exact differential equation.

Hence, line integral of y over any closed path is zero.

Also, solution is given as $\int 3x^2\,dx + \int 2x^2y\,dy = c \Rightarrow x^3 + x^2y^2 = c$

Term is M is not containing y ; x = constant

Since, $y(0) = -3$, put $x = 2$, $y = -3$ $\boxed{c = 44}$

Therefore, $x^3 + x^2y^2 = 44 \Rightarrow y = \pm\sqrt{\frac{44-x^3}{x^2}}$

Since, $y(2)$ is negative.

Hence, $y(x) = -\sqrt{\frac{44-x^3}{x^2}}$

Also, y is defined for $x \neq 0$ and $44 - x^3 \geq 0$ or $x \neq 0$ and $x \in \left\{-\sqrt[3]{44}, \sqrt[3]{44}\right\}$.

Correct options are (a), (c) and (d)

List of Exact differentials:

(i) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$ (ii) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

(iii) $d\left(\frac{y^2}{x}\right)=\frac{2xydx-y^2dx}{x^2}$ (iv) $d\left(\frac{x^2}{y}\right)=\frac{2yxdx-x^2dy}{y^2}$

(v) $d\left(\frac{y^2}{x^2}\right)=\frac{2x^2ydx-2xy^2dx}{x^4}$ (vi) $d\left(\frac{x^2}{y^2}\right)=\frac{2y^2xdx-2yx^2dy}{y^4}$

(vii) $d\left[\log(xy)\right]=\frac{xdy+ydx}{xy}$ (viii) $d(xy)=xdy+ydx$

(ix) $d\left[\tan^{-1}\frac{y}{x}\right]=\frac{xdy-ydx}{x^2+y^2}$ (x) $d\left[\tan^{-1}\frac{x}{y}\right]=\frac{ydx-xdy}{x^2+y^2}$

(xi) $d\left[\log\left(\frac{x}{y}\right)\right]=\frac{xdy-ydx}{xy}$ (xii) $d\left[\log\left(\frac{x}{y}\right)\right]=\frac{ydx-xdy}{xy}$

(xiii) $d\left[\frac{1}{2}\log\left(x^2+y^2\right)\right]=\frac{xdx+ydy}{x^2+y^2}$ (xiv) $d\left[-\frac{1}{xy}\right]=\frac{xdy+ydx}{x^2y^2}$

Example 31: Solve $ydx-xdy+\left(1+x^2\right)dx+x^2\sin ydy=0$

Soln: Dividing each term of the given equation by x^2, we get

$$\frac{ydx-xdy}{x^2}+\frac{1+x^2}{x^2}dx+\sin ydy=0 \Rightarrow -\frac{xdy-ydx}{x^2}+\left(\frac{1}{x^2}+1\right)dx+\sin ydy=0$$

$$\Rightarrow -d\left(\frac{y}{x}\right)+\left(1+\frac{1}{x^2}\right)dx+\sin ydy=0$$

Integrating, $-\left(\frac{y}{x}\right)+x-\left(\frac{1}{x}\right)-\cos y=c \Rightarrow -y+x^2-1-x\cos y=cx$

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Orthogonal Trajectories:

Two family of curves are such that every curve of either family cuts each curve of the other family at right angles, then they are called orthogonal trajectories of each other.

Example: (i) The path of a electric field is perpendicular to equipotential surface.

(ii) The line of heat flow is perpendicular to isothermal curves.

Example 32: Find the orthogonal trajectories of the family of curves xy = c

Soln: Given: xy = c $\Rightarrow y+x\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{y}{x}$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we get, $-\frac{dx}{dy}=-\frac{y}{x}$

$$\Rightarrow ydy=xdx \Rightarrow \frac{y^2}{2}=\frac{x^2}{2}+c \Rightarrow y^2-x^2=2c$$

Example 33: Find the orthogonal trajectories of the family of curves $x^2 + y^2 = C$

Soln: Given: $x^2 + y^2 = C \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we get, $-\frac{dx}{dy} = -\frac{x}{y} \Rightarrow \int\frac{dx}{x} = \int\frac{dy}{y} \Rightarrow \ln x + \ln m = \ln y \Rightarrow y = mx$

Example 34: If y(x) satisfies the differential equation $\frac{dy}{dx} + 2y = 2 + e^{-x^2}$ with y(0) = 0, then $\underset{x\to\infty}{Lt}\, y(x)$ is equal to

(a) 0 (b) 1 (c) 2 (d) -1

Soln: Given differential equation is a first order linear differential equation.

I.F. = $e^{\int 2dx} = e^{2x}$;

Therefore the solution is

$$y.e^{2x} = \int\left(2e^{2x} + e^{2x-x^2}\right)dx \Rightarrow y.e^{2x} = e^{2x} + \int e^{2x-x^2}dx \Rightarrow y = 1 + e^{-2x}\int e^{2x-x^2}dx \Rightarrow \underset{x\to\infty}{Lt}\, y(x) = 1$$

3.3 Linear Second Order Differential Equations

CASE I: Homogeneous Differential equation:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + QY = 0 \qquad (P, Q \text{ are constant coefficients})$$

Say, $y = ce^{mx}$ be the trial solution

$$\Rightarrow ce^{mx}(m^2 + Pm + Q) = 0 \Rightarrow (m^2 + Pm + Q) = 0$$

The above equation is known as auxilliary equation corresponding to the differential equation.

Suppose m_1, m_2 are two solutions of auxillary equation. Depending on m_1, m_2, there will be varoius forms of solution possible. Now,

(i) If m_1, m_2 are real and unequal, then solution will be of the form:

$$y = c_1e^{m_1x} + c_2e^{m_2x}$$

(ii) If m_1, m_2 are real and equal ($m_1 = m_2 = m$), then solution will be of the form:

$$y = (c_1 + c_2x)e^{mx}$$

(iii) If m_1, m_2 are complex numbers, then solution will be of the form:

$$y = c_1e^{m_1x} + c_2e^{m_2x}$$

(iv) If m_1, m_2 are complex conjugate of each other i.e. m_1, $m_2 = \alpha \pm i\beta$, then solution will be of the form:

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

Example 35: Solve $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

Soln: The given differential equation is $(D^2-7D+12)y=0$ [where $D\equiv\frac{d}{dx}$]

The auxilary equation is $m^2-7m+12=0 \quad \Rightarrow (m-3)(m-4)=0 \Rightarrow m=3,4$

The complete solution is $y(x)=C_1e^{3x}+C_2e^{4x}$, where C_1 & C_2 are constants.

Example 36: Solve $\frac{d^3y}{dx^3}-8y=0$

Soln: The auxilary equation is $m^3-8=0 \Rightarrow (m-2)(m^2+2m+4)=0 \Rightarrow m=2,-1+i\sqrt{3},-1-i\sqrt{3}$

The complete solution is $y(x)=C_1e^{2x}+e^{-x}\left[C_2\cos\sqrt{3}\,x+C_3\sin\sqrt{3}\,x\right]$

Example 37: Consider the differential equation.

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0; \quad at\ t=0,\ x=1\ \&\ \frac{dx}{dt}=0$$

Find the value of x at $t=1$.

Soln: The auxilary equation is $m^2+2m+1=0 \Rightarrow m=-1,-1$

Therefore, the solution is $x(t)=C_1e^{-t}+C_2te^{-t} \Rightarrow \frac{dx}{dt}=-C_1e^{-t}+C_2e^{-t}-C_2t\,e^{-t}$

At $t=0, x=1 \Rightarrow C_1=1$

At $t=0$, $\frac{dx}{dt}=0 \Rightarrow 0=-C_1+C_2 \Rightarrow C_1=C_2=1 \Rightarrow x(t)=e^{-t}+t\,e^{-t}$

At t = 1, the value of x will be $e^{-1}+e^{-1}=\frac{2}{e}$

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Example 38: Solution of the differential equation $\frac{d^2y}{dx^2}+2p\frac{dy}{dx}+(p^2+q^2)y=0$ is

(a) $y=e^{px}(c_1\cos qx+c_2\sin qx)$ (b) $y=e^{-px}(c_1+c_2x)$

(c) $y=e^{-px}(c_1\cos qx+c_2\sin qx)$ (d) None of these

Soln: Auxillary equation is $\{m^2+2pm+(p^2+q^2)\}y=0$

For non-trivial solution,

$$m^2+(p+q)m+(p-q)m+(p^2+q^2)=0$$

$$\Rightarrow m[m+(p+q)]+(p-q)[m+(p+q)]=0$$

$$\Rightarrow [m+(p+q)][m+(p-q)]=0$$

Hence, $m=-(p+q),-(p-q)$

Therefore, C.F. $y=c_1e^{-(p+q)x}+c_2e^{-(p-q)x}=e^{-px}\left[c_1e^{-qx}+c_2e^{qx}\right]$

Since, $\cosh qx = \frac{e^{qx}+e^{-qx}}{2}$, $\sinh qx = \frac{e^{qx}-e^{-qx}}{2}$

On putting the values of e^{qx} and e^{-qx}, we get

$y(x) = e^{-px}\left[c_1 \cosh qx + c_2 \sinh qx\right]$

Correct option is (d)

Example 39: Value of $y(x=-1)$, where $y(x)$ is the solution of the differential equation $y'' + 2y' - 8y = 0$ with $y(0) = 3$, $y'(0) = -12$, is _____.

Soln: Auxillary equation,

$$(m^2 + 2m - 8)\, y = 0 \Rightarrow m^2 + 2m - 8 = 0 \Rightarrow (m+4)(m-2) = 0 \Rightarrow m = -4, 2$$

Therefore, C.F. $y(x) = Ae^{-4x} + Be^{2x}$

Since, $y(0) = 3$, put $x = 0$, $y = 3$, we get $3 = A + B$ ... (1)

and $y'(x) = -4Ae^{-4x} + 2Be^{2x}$

Therefore, $y'(x=0) = -12 \Rightarrow -4A + 2B = -12$ ... (2)

Solving eqn. (1) and eqn. (2); $A = 3$, $B = 0$

Therefore, $y(x) = 3e^{-4x} \Rightarrow y(x=-1) = 3e^{4} = 161.80$.

Correct answer is (161.80)

Example 40: A function $f(x)$ obeys the differential equation $\frac{d^2f}{dx^2} - (3-4i)f = 0$ and satisfies the conditions $f(0) = 1$ and $f(x) \to 0$ as $x \to \infty$. The value of $f(\pi)$ is:

(a) $e^{2\pi}$ (b) $e^{-2\pi}$ (c) $-e^{-2\pi}$ (d) $-e^{2\pi i}$

Soln: Given differential equation: $\frac{d^2f}{dx^2} - (3-4i)f = 0$

Let, $f = c.e^{mx}$ be trial solution.

$$\Rightarrow \quad \frac{df}{dx} = c.m\, e^{mx} \quad \Rightarrow \quad \frac{d^2f}{dx^2} = c.m^2\, e^{mx}$$

Substituting in the given equation, we get

$$m^2 - (3-4i) = 0 \quad (e^{mx} \neq 0)$$

$$\Rightarrow \quad m = \sqrt{3-4i} = \sqrt{4-1-4i} = \sqrt{(2)^2 + (i)^2 - 2.2.i} = \sqrt{(2-i)^2} = \pm(2-i)$$

Therefore, $f(x) = A\, e^{(2-i)x} + B\, e^{-(2-i)x}$

Since, $f(0) = 1 \quad \Rightarrow A + B = 1$

Since, $f(x) \to 0$ when $x \to \infty \Rightarrow A = 0$, then $B = 1$

$$\therefore \quad f(x) = e^{-(2-i)x} \quad \Rightarrow f(\pi) = e^{-(2-i)\pi} = e^{-2\pi}\, e^{i\pi} \quad \Rightarrow \boxed{f(\pi) = -e^{-2\pi}}$$

Correct answer is (c)

Example 41: Consider the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$$

with the initial conditions $x(0)=0$ and $\dot{x}(0)=1$. The solution $x(t)$ attains its maximum value when 't' is

(a) 1/2 (b) 1 (c) 2 (d) ∞

Soln: $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$

Assume the trial solution : $x=ce^{mt}$

$\Rightarrow (m^2+2m+1)c.e^{mt}=0 \Rightarrow (m^2+2m+1)=0 \Rightarrow (m+1)^2=0 \Rightarrow m=-1,-1$

$\therefore \quad x(t)=(c_1+c_2t)e^{-t}$

Putting $x(0)=0 \Rightarrow c_1=0$

$\therefore \quad x(t)=c_2t\,e^{-t} \Rightarrow \dot{x}(t)=\frac{dx}{dt}=c_2e^{-t}-c_2te^{-t}$

putting $\dot{x}(0)=1 \Rightarrow c_2=1$

$\therefore \quad x(t)=t\,e^{-t} \Rightarrow \frac{dx}{dt}=e^{-t}-te^{-t} \Rightarrow \frac{d^2x}{dt^2}=te^{-t}-2e^{-t}$

For maxima or minima, $\frac{dx}{dt}=0 \Rightarrow t=1$

At $t=1, \frac{d^2x}{dt^2}=-e^{-1}=-ve$

Therefore, $x(t)$ attains its maximum value at $t=1$

Correct answer is (b)

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Example 42: Consider the differential equation $\frac{d^2x}{dt^2}+\frac{2dx}{dt}+x=0$. At time $t=0$, it is given that $x=1$ and $\frac{dx}{dt}=0$. At $t=1$, the value of x is given by

(a) $\frac{1}{e}$ (b) $\frac{2}{e}$ (c) 1 (d) $\frac{3}{e}$

Soln: $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$

Let, $x=C.e^{mt}$ be the trial solution.

So, $(m^2+2m+1)C.e^{mt}=0 \Rightarrow m=-1,-1$

Therefore, $x=(C_1+C_2t)e^{-t}$

At $t=0, x=1$ then $C_1=1$; so, $x=(1+C_2t)e^{-t}$

Now, $\frac{dx}{dt}=C_2e^{-t}+(1+C_2t)(-e^{-t})$

At $t=0, \frac{dx}{dt}=0$, then $C_2=-1=0 \Rightarrow C_2=1$

So, $C_2x=(1+t)e^{-t}$; At $t=1, x=\frac{2}{e}$

Correct option is (b)

Example 43: The solution of the differential equation $\frac{d^2y}{dt^2}-y=0$ subject to the boundary conditions $y(0)=1$ and $y(\infty)=0$, is

(a) $\cos t+\sin t$ (b) $\cosh t+\sinh t$ (c) $\cos t-\sin t$ (d) $\cosh t-\sinh t$

Soln: $\frac{d^2y}{dt^2}-y=0$

Assume, the trial solution to be $y=c\cdot e^{mx}$

$\Rightarrow m^2-1=0 \Rightarrow m=\pm 1$

So, $y=Ae^{t}+Be^{-t}$

Putting $y(t=0)=1$, we get $A+B=1$

Putting $y(t=\infty)=0$, we get $A=0, B=1$

Therefore, $y(t)=e^{-t}=\cosh t-\sinh t$

Correct option is (d)

CASE II: Non-Homogeneous Differential equation:

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R(x)$$

The above equation can be written as

$$(D^2+P.D+Q)y=R(x) \Rightarrow f(D)y=R(x) \quad \left[\text{where } D\equiv\frac{d}{dx}\right]$$

The complete solution of the above differential equation consists of two parts:

(i) Complementary function (C.F.): same as the solution of the equivalent homogenous differential equation,

(ii) Particular integral (P.I): depends on the form of $R(x)$

Rules to get Particular Integral:

(1) $R(x)=e^{ax}$, then particular integral $=\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$

(i) If $f(a)=0$, then particular integral $=x\frac{1}{f'(a)}e^{ax}$

(ii) If $f'(a)=0$ then particular integral $=x^2\frac{1}{f''(a)}e^{ax}$

(2) $R(x)$ = polynomial of x; then particular integral = $\frac{1}{f(D)}x^n$ [Expand $\frac{1}{f(D)}$ in series containing poisitive powers of D]

(3) $R(x) = \cos ax$ *or* $\sin ax$; then particular integral = $\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$;

(i) if $f(-a^2) = 0$, then particular integral = $\frac{x}{f'(-a^2)}\sin ax$

(4) $R(x) = e^{ax}\phi(x)$; then particular integral = $\frac{1}{f(D)}e^{ax}\cdot\phi(x) = e^{ax}\frac{1}{f(D+a)}\phi(x)$

Example 44: Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$

Soln: Auxillary equation: $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2 \Rightarrow$ C.F. $= C_1e^x + C_2e^{2x}$

Particular integral (P.I) $= \frac{e^{5x}}{D^2 - 3D + 2} = \frac{e^{5x}}{5^2 - 3.5 + 2} = \frac{e^{5x}}{12}$

Therefore, Complete solution is $y = C.F. + P.I. = C_1e^x + C_2e^{2x} + \frac{e^{5x}}{12}$

Example 45: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = \sin 2x$

Soln: Auxillary equation is $m^2 - m - 2 = 0 \Rightarrow m = -1, 2 \Rightarrow C.F. = C_1e^{-x} + C_2e^{2x}$

$$P.I. = \frac{\sin 2x}{D^2 - D - 2} = \frac{\sin 2x}{-(2)^2 - D - 2} = \frac{-\sin 2x}{D + 6}$$

$$= -\frac{(D-6)\sin 2x}{D^2 - 36} = -\frac{(D-6)\sin 2x}{-(2)^2 - 36} = \frac{1}{40}(D-6)\sin 2x = \frac{2\cos 2x - 6\sin 2x}{40}$$

Therefore, Complete solution is $y = C.F. + P.I. = C_1e^{-x} + C_2e^{2x} + \frac{1}{20}(\cos 2x - 3\sin 2x)$

Example 46: Solve $(D^2 - 2D + 1)y = x^2e^{3x}$

Soln: Auxillary equation is $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1 \Rightarrow C.F. = C_1e^x + C_2xe^x = (C_1 + C_2x)e^x$

$$P.I. = \frac{1}{D^2 - 2D + 1}(x^2e^{3x}) = \frac{1}{(D-1)^2}e^{3x}x^2 = e^{3x}\frac{1}{\{(D+3)-1\}^2}x^2 = e^{3x}.\frac{1}{(D+2)^2}x^2$$

$$= \frac{1}{4}e^{3x}\left(1 + \frac{D}{2}\right)^{-2}x^2 = \frac{1}{4}e^{3x}\left[1 - D + \frac{3}{4}D^2 + \ldots.\right]x^2 \quad \Rightarrow \quad P.I. = \frac{e^{3x}}{4}\left[x^2 - 2x + \frac{3}{2}\right]$$

Hence complete solution is y = C.F. + P.I. $\Rightarrow y = (C_1 + C_2x)e^x + \frac{e^{3x}}{4}\left[x^2 - 2x + \frac{3}{2}\right]$

Example 47: Solve $\left(D^3-D^2-6D\right)y=x^2+1$

Soln: Auxiliary equation: $m\left(m^2-m-6\right)=0 \Rightarrow$ m = 0,3,-2 $\Rightarrow$ $C.F.=c_1+c_2e^{3x}+c_3e^{-2x}$

$$P.I.=\frac{1}{D^3-D^2-6D}\left(x^2+1\right)=\frac{1}{-6D\left(1+\frac{D}{6}-\frac{D^2}{6}\right)}\left(x^2+1\right)=-\frac{1}{6D}\left[1+\left(\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1}\left(x^2+1\right)$$

$$=-\frac{1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\left(\frac{D}{6}-\frac{D^2}{6}\right)^2-.....\right]\left(x^2+1\right)$$

$$=-\frac{1}{6D}\left[x^2+1-\frac{1}{6}D\left(x^2-1\right)+\frac{7}{36}D^2\left(x^2+1\right)+.....\right]=-\frac{1}{6D}\left[x^2+1-\frac{1}{6}(2x+0)+\frac{7}{36}(2+0)+...\right]$$

$$=-\frac{1}{6D}\left[x^2-\frac{1}{3}x+\frac{25}{18}\right]=-\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

Hence, the required solution is $y=C.F.+P.I=y=c_1+c_2e^{3x}+c_3e^{-2x}-\left(\frac{1}{18}\right)\left(x^3-\frac{x^2}{2}+\frac{25x}{6}\right)$

Example 48: Solve $\left(D^3+8\right)y=x^4+2x+1$

Soln: The auxialiary equation is $(m+2)\left(m^2-2m+4\right)=0 \Rightarrow m=-2,1\pm i\sqrt{3}$

$$\Rightarrow C.F.=c_1e^{-2x}+e^x\left(c_2\cos\sqrt{3}x+c_3\sin\sqrt{3}x\right)$$

$$P.I.=\frac{1}{D^3+8}\left(x^4+2x+1\right)=\frac{1}{8\left(1+\frac{1}{8}D^3\right)}\left(x^4+2x+1\right)$$

$$=\frac{1}{8}\left(1+\frac{1}{8}D^3\right)^{-1}\left(x^4+2x+1\right)=\frac{1}{8}\left(1-\frac{1}{8}D^3+....\right)\left(x^4+2x+1\right)$$

$$=\frac{1}{8}\left[\left(x^4+2x+1\right)-\frac{1}{8}D^3\left(x^4+2x+1\right)\right] \qquad =\frac{1}{8}\left[\left(x^4+2x+1\right)-3x\right]=\frac{1}{8}\left(x^4-x+1\right)$$

$$\Rightarrow y=c_1e^{-2x}+e^x\left(c_2\cos\sqrt{3}x+c_3\sin\sqrt{3}x\right)+\left(\frac{1}{8}\right)\left(x^4-x+1\right)$$

Example 49: Solve $\left(D^2-1\right)y=\cosh x\cos x=\frac{1}{2}\left(e^x+e^{-x}\right)\cos x$

Soln: Auxialiary equation is $m^2-1=0 \Rightarrow$ m = -1,1 $\Rightarrow$ C.F. = $c_1e^x+c_2e^{-x}$

$$\text{P.I.}=\frac{1}{2}\frac{1}{D^2-1}e^x\cos x+\frac{1}{2}\frac{1}{D^2-1}e^{-x}\cos x=\frac{1}{2}e^x\frac{1}{(D+1)^2-1}\cos x+\frac{1}{2}e^{-x}\frac{1}{(D-1)^2-1}\cos x$$

$$=\frac{1}{2}e^{x}\frac{1}{D^2+2D}\cos x+\frac{e^{-x}}{2}\frac{1}{D^2-2D}\cos x=\frac{1}{2}e^{x}\frac{1}{-1+2D}\cos x+\frac{e^{-x}}{2}\frac{1}{-1-2D}\cos x$$

$$=\frac{e^{x}}{2}(2D+1).\frac{1}{(2D+1)(2D-1)}\cos x-\frac{e^{-x}}{2}(2D-1)\frac{1}{(2D-1)(2D+1)}\cos x$$

$$=\frac{e^{x}}{2}(2D+1)\frac{1}{(4D^2-1)}\cos x-\frac{e^{-x}}{10}(2D-1)\frac{1}{4D^2-1}\cos x$$

$$=\frac{e^{x}}{2}(2D+1)\frac{1}{-4-1}\cos x-\frac{e^{-x}}{2}(2D-1)\frac{1}{-4-1}\cos x$$

$$=-\frac{e^{x}}{10}(2D+1)\cos x+\frac{e^{-x}}{10}(2D-1)\cos x=-\frac{e^{x}}{10}(-2\sin x+\cos x)+\frac{e^{-x}}{10}(-2\sin x-\cos x)$$

$$\therefore\quad y=c_1e^{x}+c_2e^{-x}-\left(\frac{1}{10}\right)e^{x}(-2\sin x+\cos x)+\left(\frac{1}{10}\right)e^{-x}(-2\sin x-\cos x)$$

$$\Rightarrow y=c_1e^{x}+c_2e^{-x}+\frac{2}{5}\sin x\left(\frac{e^{x}-e^{-x}}{2}\right)-\frac{1}{5}\cos x\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$\Rightarrow y=c_1e^{x}+c_2e^{-x}+\left(\frac{2}{5}\right)\sin x\sinh x-\left(\frac{1}{5}\right)\cos x\cos hx$$

Example 50: Solve $\left(D^2+2D+1\right)y=x\cos x$

Soln: Auxiliary equation is $(m+1)^2=0\Rightarrow m=-1,-1$

$$P.I.=\frac{1}{(D+1)^2}x\cos x=\text{real part of }\frac{1}{(D+1)^2}xe^{ix}$$

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Now, $$\frac{1}{(D+1)^2}xe^{ix}=e^{ix}\frac{1}{(D+i+1)^2}x=\frac{e^{ix}}{(1+i)^2}\frac{1}{\left[1+\frac{D}{1+i}\right]^2}.x$$

$$=\frac{e^{ix}}{2i}\left[1+\frac{D}{1+i}\right]^{-2}x=\frac{e^{ix}}{2i}\left[1-\frac{2D}{1+i}+.....\right]x=-\frac{1}{2}ie^{ix}\left[1-\frac{1-i}{2}.2\right]$$

$$=-\frac{1}{2}ie^{ix}\left[(x-1)+i\right]=\frac{1}{2}e^{ix}\left[(x-1)i-1\right]=-\frac{1}{2}(\cos x+i\sin x)\left[(x-1)i-1\right]$$

$$\text{P.I.}=-\frac{1}{2}\left[-\cos x-(x-1)\sin x\right]$$

$$\therefore\qquad y=(c_1+c_2x)e^{-x}+\left(\frac{1}{2}\right)\left[\cos x+(x-1)\sin x\right]$$

Example 51: Solution of the differential equation $\frac{d^3y}{dx^3}-5\frac{d^2y}{dx^2}+7\frac{dy}{dx}-3y=e^{2x}\cosh x$ is

(a) $(c_1+c_2x)e^x+\frac{x}{8}e^{3x}-\frac{x^2}{8}e^x+c_3e^{3x}$ (b) $2c_1e^x+c_2e^{3x}+\frac{x}{8}e^{3x}-\frac{x}{8}e^{3x}$

(c) $c_1e^x+c_2e^{3x}+\frac{x}{8}e^{3x}$ (d) $(c_1+c_2x)e^x+c_3e^{3x}+\frac{x}{8}e^{3x}-\frac{x}{8}e^x$

Soln: Auxillary equation $(m^3-5m^2+7m-3)y=0$

For non-trivial solution, $m^2-5m^2+7m-3=0\Rightarrow(m-1)(m^2-4m+3)=0$

$\Rightarrow(m-1)(m-3)(m-1)=1\Rightarrow m=1,1,3.$

Therefore, C.F. $(c_1+c_2x)e^x+c_3e^{3x}$

Now, P.I. $\frac{1}{D^3-5D^2+7D-3}\times e^{2x}\cosh x=\frac{1}{D^3-5D^2+7D-3}\times e^{2x}\times\frac{e^x+e^{-x}}{2}$

$\Rightarrow\frac{1}{2}\frac{1}{D^3-5D^2+7D-3}e^{3x}+\frac{1}{2}\frac{1}{D^3-5D^2+7D-3}\times e^x$

$=\frac{1}{2}\frac{x}{3D^2-10D+7}e^{3x}+\frac{1}{2}\frac{x}{3D^2-10D+7}e^x$

$=\frac{x}{8}e^{3x}+\frac{1}{2}\frac{x^2}{6D-10}e^x$ $=\frac{x}{8}e^{3x}-\frac{x^2}{8}e^x$

Therefore, $y(x)=(c_1+c_2x)e^x+c_3e^{3x}+\frac{x}{8}e^{3x}-\frac{x^2}{8}e^x$.

Correct option is (a)

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Example 52: Particular integral of the differential equation $(D^3+8)y=x^4+x+1$, where $D=\frac{d}{dx}$ is

(a) $\frac{1}{8}(x^4+x+1)$ (b) $\frac{1}{8}(x^3+4x+1)$ (c) $\frac{1}{8}(x^4+x-1)$ (d) $\frac{1}{8}(x^4+4x+1)$

Soln: P.I. $=\frac{1}{D^3+8}(x^4+x+1)=\frac{1}{8}\left(1+\frac{D^3}{8}\right)^{-1}(x^4+x+1)$

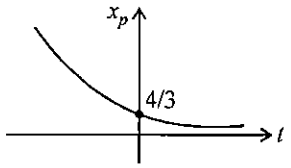
$=\frac{1}{8}\left(1+\frac{D^3}{8}+\cdots\right)(x^4+x+1)$

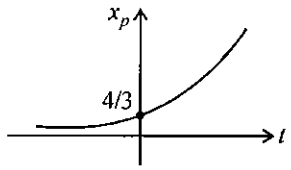
$=\frac{1}{8}(x^4+x+1+\frac{1}{8}\times24x)$

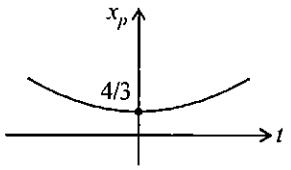
$=\frac{1}{8}(x^4+4x+1).$

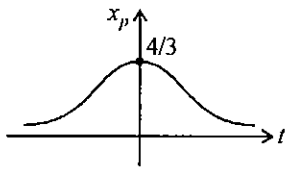
Correct option is (d)

Example 53: If x_p is the particular integral of second order differential equation $\frac{d^2x}{dt^2} - x = 4e^{-2t}$, then which of the following graph best represent the x_p with t?

(a) 

(b) 

(c) 

(d) 

Soln: $\text{P.I.} = \frac{1}{D^2-1} 4e^{-2t} = \frac{1}{4-1} \times 4e^{-2t}$

Hence, P.I. $x_p = \frac{4}{3} e^{-2t}$.

Correct option is (a)

Example 54: Which of the following statements is/are correct about the differential equation $(D^2 + 2D + 1)y = x\cos x$

(a) C.F. of this equation is $(c_1 + c_2 x)e^{-x}$

(b) C.F. for this equation is $c_1 e^{-x} + c_2 e^{2x}$

(c) P.I. for this equation is $\frac{1}{2}[(x-1)\sin x + \cos x]$

(d) P.I. for this equation is $\frac{1}{2}[(x-1)\sin x - \cos x]$

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Soln: Auxillary equation $(m^2 + 2m + 1)y = 0$

For non trivial solution $m^2 + 2m + 1 = 0 \Rightarrow \boxed{m = -1, -1}$

Therefore, C.F. $= (c_1 + c_2 x)e^{-x}$

$$\text{P.I.} = \frac{1}{(D+1)^2} x\cos x \Rightarrow \text{R.P. of } \frac{1}{(D+1)^2} x e^{ix}$$

$$= e^{ix} \frac{1}{(D+i+1)^2} x = \frac{e^{ix}}{(1+i)^2} \frac{1}{\left[1+\frac{D}{1+i}\right]^2} x$$

$$= \frac{e^{ix}}{2i}\left[1+\frac{D}{1+i}\right]^{-2} x = \frac{e^{ix}}{2i}\left[1-\frac{2D}{1+i}\right]x$$

$$=\frac{e^{ix}}{2i}\left[x-\frac{2}{1+i}\times\frac{1-i}{1-i}\right]=\frac{e^{ix}}{2i}\left[x-(1-i)\right]$$

Hence real part $\frac{1}{2}\left[(x-1)\sin x+\cos x\right]$

Correct options are (a) and (c)

Example 55: If $x(t)$ and $y(t)$ are differential functions on R such that $x(0)=2$ and $y(0)=1$. Also $x(t)+y'(t)=1$ and $x'(t)+y(t)=1\ \forall\ t\in[0,1]$, then the value of $x(\log 3)$ is _____.

Soln: We have $x(t)+y'(t)=1$ ... (1)

and $x'(t)+y(t)=1$ ... (2)

Hence, from Eq. (1) and Eq. (2) we get

$x''(t)+1-x(t)=0\ \Rightarrow x''(t)-x(t)=1$

Auxillary equation $(m^2-1)\,x(t)=0$

For non trivial solution, $m^2-1=0\Rightarrow m=\pm1$

Therefore, C.F. = $Ae^{t}+Be^{-t}$

Now, P.I. = $\frac{1}{D^2-1}1=\frac{1}{D^2-1}e^{0}=-1$

Hence, $x(t)=Ae^{t}+Be^{-t}-1$

Since, $x(0)=2$, put $t=0$, $x=2$

$\Rightarrow 2=A+B-1\Rightarrow A+B=3$ ... (3)

and $y(0)=1$, hence from Eq. (2); $x'(t=0)=0$

Here, $x'(t)=Ae^{t}-Be^{-t}$

Therefore, $x'(t=0)=A-B=0\Rightarrow A=B$ ... (4)

Therefore, from Eq. (3) and Eq. (4), we get $A=B=\frac{3}{2}$

Therefore, $x(t)=\frac{3}{2}\left(e^{t}+e^{-t}\right)-1$

$$\Rightarrow x(\log 3)=\frac{3}{2}\left(e^{\log 3}+e^{-\log 3}\right)-1=\frac{3}{2}\left(3+\frac{1}{3}\right)-1=\frac{3}{2}\times\frac{10}{3}-1=4.$$

Correct answer is (4)

Example 56: Value of the particular integral of the differential equation

$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+8y=3\cos x$ at $x=\frac{\pi}{2}$ is_____.

Soln: Particular integral = $\frac{1}{D^2-6D+8}3\cos x$ (put $D^2=-1$)

$$=3\times\frac{1}{-1-6D+8}\cos x=-3\times\frac{1}{6D-7}\cos x=-3\times\frac{1}{6D-7}\times\left(\frac{6D+7}{6D+7}\right)\cos x$$

$$= -3 \times \frac{(6D+7)}{36D^2-49}\cos x = \frac{-3\times(-6\sin x+7\cos x)}{36-49}$$

Hence, P.I. at $x=\frac{\pi}{2} = \frac{-3(-6+0)}{-13} = \frac{-18}{13} = 1.38.$

Correct answer is (1.38)

Example 57: If y_1 and y_2 are the particular solution of the differential equation $y''-y'-2y=4x^2$ and $y''-y'-2y=\sin 2x$ respectively, then value of y_1/y_2 at $x=0$ is _____.

Soln: Particular solution of 1st differential equation,

$$y_1 = \frac{1}{D^2-D-2}4x^2 = -\frac{1}{2}\left(1-\frac{D^2-D}{2}\right)^{-1}(4x^2)$$

$$= -\frac{1}{2}\left(1+\frac{D^2-D}{2}+\left(\frac{D^2-D}{2}\right)^2\cdots\right)(4x^2) = -2x^2+2x-3$$

Particular solution of 2nd differential equation,

$$y_2 = \frac{1}{D^2-D-2}\sin 2x, \text{ (put } D^2=-4)$$

$$= \frac{1}{-D-6}\sin 2x = -\frac{1}{D+6}\times\frac{D-6}{D-6}\sin 2x$$

$$= \frac{(D-6)\sin 2x}{D^2-36} = -\frac{2\cos 2x-6\sin 2x}{-40} = -\frac{3}{20}\sin 2x+\frac{1}{20}\cos 2x$$

Therefore, $\left(\frac{y_1}{y_2}\right)_{x=0} = -\frac{3}{1/20} = -60.$

Correct answer is (- 60)

CAREER ENDEAVOUR

Example 58: The solution of the differential equation for $y(t)$: $\frac{d^2y}{dt^2}-y=2\cosh(t)$, subject to the initial conditions $y(0)=0$ and $\left.\frac{dy}{dt}\right|_{t=0}=0$ is

(a) $\frac{1}{2}\cosh(t)+t\sinh(t)$ (b) $-\sinh(t)+t\cosh(t)$

(c) $t\cosh(t)$ (d) $t\sinh(t)$

Soln: $\frac{d^2y}{dt^2}-y=2\cdot\cosh t$

$C.F. = Ae^t + Be^{-t}$

$$P.I. = \frac{1}{D^2-1}(2\cdot\cosh t) = \frac{1}{(D^2-1)}(e^t+e^{-t})$$

$$= t\frac{1}{2D}(e^t+e^{-t}) = \frac{t}{2}(e^t-e^{-t}) = t\sinh t$$

Total solution, $y = Ae^{t} + Be^{-t} + t\sinh t$

Putting the condition, $\left.\frac{dy}{dt}\right|_{t=0} = 0 \Rightarrow A - B = 0$

Solving above two equations, $A = 0, B = 0$

Therefore, $y = t\sin t$

Correct option is (d)

Wronskian of the differential equation:

If $y_1(x)$ and $y_2(x)$ are two soultions of a second order linear homogeneous differential equation, then Wronskian of the differential equation is defined as

$$W(y_1, y_2, x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

(i) if $W(y_1, y_2, x) = 0$, then $y_1(x), y_2(x)$ are linearly dependent.
(ii) if $W(y_1, y_2, x) \neq 0$, then $y_1(x), y_2(x)$ are linearly independent.

Note: (i) If y_1 and y_2 are two linearly independent solution of a second order linear homogeneous differential equation then their linear combination $c_1y_1 + c_2y_2$ will also be a solution of that differential equation.

(ii) The value of the Wronskian at different values of x can be related by the following relation:

$$W(x = x_1) = W(x = x_0)e^{-\int_{x_0}^{x_1} P(x)dx}$$

(iii) A differential equation is said to have dimension 'k' if the differential equation has 'k' number of linearly independent solutions.

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Cauchy-Euler differential equation of second order:

$$a_2x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_0y = Q(x)$$

Put $x = e^z \Rightarrow \ln x = z \Rightarrow \frac{1}{x} = \frac{dz}{dx}$

$$\frac{dy}{dx} = \frac{dy}{dx}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} \Rightarrow x^2\frac{dy}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2}$$

Then the given equation becomes

$$a_2\frac{d^2y}{dz^2} + (a_1 - a_2)\frac{dy}{dz} + a_0y = Q(z)$$

The above equation is nothing but a linear second order non-homogeneous differential equation.

Example 59: $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4$

Soln: $x=e^z \Rightarrow \log x=z \Rightarrow \frac{1}{x}=\frac{dz}{dx}$

Putting $x\frac{dy}{dx}=\frac{dy}{dz}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$, we get $\Rightarrow \frac{d^2y}{dz^2}-3\frac{dy}{dz}-4y=e^{4z}$

Auxillary equation: $m^2-3m-4=0 \Rightarrow m=4,-1 \Rightarrow$ C.F. $=c_1e^{4z}+c_2e^{-z}$

Particular integral $=\frac{1}{D^2-3D-4}e^{4z}=z\cdot\frac{1}{2D-3}e^{4z}=\frac{z}{5}e^{4z}$

Total solution is $c_1e^{4z}+c_2e^{-z}+\frac{z}{5}e^{4z}=c_1x^4+c_2\frac{1}{x}+\frac{1}{5}x^4\log x$

Example 60: $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-20y=(x+1)^2$

Soln: $x=e^z \Rightarrow \log x=z \Rightarrow \frac{1}{x}=\frac{dz}{dx}$

Putting $x\frac{dy}{dx}=\frac{dy}{dz}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$, we get $\Rightarrow \frac{d^2y}{dz^2}+\frac{dy}{dz}-20y=(e^z+1)^2$

$C.F.=c_1e^{4z}+c_2e^{-5z}=c_1x^4+c_2x^{-5}$

$P.I.=\frac{1}{D^2+D-20}(e^z+1)^2=\frac{1}{D^2+D-20}(e^{2z}+2e^z+1)=-\frac{1}{14}e^{2z}-\frac{1}{9}e^z-\frac{1}{20}$

Total solution $y=c_1x^4+c_2x^{-5}-\frac{1}{14}x^2-\frac{1}{9}x-\frac{1}{20}$

Example 61: A function $y(z)$ satisfies the ordinary differential equation $y''+\frac{1}{z}y'-\frac{m^2}{z^2}y=0$, where $m=0,1,2,3,....$ Consider the four statements P, Q, R, S as given below.

P : z^m and z^{-m} are linearly independent solutions for all values of m

Q : z^m and z^{-m} are linearly independent solutions for all values of $m>0$

R : $\ell n\ z$ and 1 are linearly independent solutions for $m=0$

S : z^m and $\ell n\ z$ are linearly independent solutions for all values of m

The correct option for the combination of valid statement is

(a) P, R and S only (b) P and R only (c) Q and R only (d) R and S only

Soln: $y''+\frac{1}{z}y'-\frac{m^2}{z^2}y=0 \Rightarrow \quad z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}-m^2y=0 \rightarrow$ Cauchy Euler equation.

Putting $z=e^x$ and replacing

$$z\frac{dy}{dz} \text{ by } \frac{dy}{dx} \text{ and } z^2\frac{d^2y}{dz^2} \text{ by } \frac{d^2y}{dx^2}-\frac{dy}{dx}$$

We get,

$$\frac{d^2y}{dx^2}-\frac{dy}{dx}+\frac{dy}{dx}-m^2y=0$$

Trial soln : $y=C\cdot e^{kx}$

$\Rightarrow \quad \left(k^2-m^2\right)=0 \Rightarrow k=\pm m$

So, $\quad y=C_1e^{mx}+C_2e^{-mx}=C_1z^m+C_1z^{-m}$

Therefore, z^m and z^{-m} are linearly independent solutions for $m>0$.

For $m=0, k=0 \Rightarrow y=\left(C_1+C_2x\right)e^{0.x}=\left(C_1+C_2\ell nz\right)$

Therefore, 1 and $l$nz are linearly independent solutions for $m=0$

Correct option is (c)

Method of Variation of parameters:

Consider the following second order non-homogeneous linear differential equation:

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R(x)$$

Suppose $y_1=e^{m_1x}$ and $y_2=e^{m_2x}$ are two linearly independent solutions of the given differential equation, then

Complementary function = C.F. = Ay_1+By_2

Particular integral = P.I. = uy_1+vy_2

where $u=\int-\frac{y_2R(x)}{y_1y_2'-y_2y_1'}dx$ and $v=\int\frac{y_1R(x)}{y_1y_2'-y_2y_1'}dx$

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Example 62: Solve: $\frac{d^2y}{dx^2}+y=\tan x$

Soln: Auxilliary equation: $m^2+1=0 \Rightarrow m=\pm i \Rightarrow C.F.=A\cos x+B\sin x$

In this case, $y_1=\cos x, y_2=\sin x$ and $y_1y_2'-y_1'y_2=1$

Particular solution = $u\cos x+v\sin x$

$$u=-\int y_2\tan x\,dx=-\int\frac{\sin^2x}{\cos x}dx=-\int\frac{1-\cos^2x}{\cos x}dx$$

$$=-\int(\sec x-\cos x)dx=-\ln(\sec x+\tan x)+\sin x$$

$$v=\int y_1\tan x\,dx=\int\sin x\,dx=-\cos x$$

Particular solution = $-\cos x.\ln(\sec x+\tan x)$

Total Solution is $y = A\cos x + B\sin x - \cos x.\ln(\sec x + \tan x)$

3.4 Special Functions

[Note: Question from this topic is only asked in BHU entrance examination]

(A) Legendre Differential Equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

where n is a positive real integer.

Note: $x = \pm 1$ are regular singular points of the Legendre differential equation.

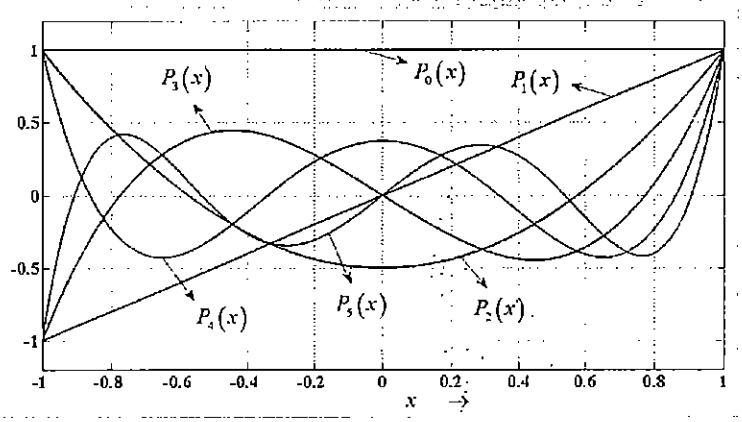
Solution:

Legendre polynomial of degree n and it is given by Rodrigue's formula as follows:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

The first few Legendre polynomials are

$$P_0(x) = 1; \qquad P_1(x) = x; \qquad P_2(x) = \frac{1}{2}(3x^2-1); \qquad P_3(x) = \frac{1}{2}(5x^3-3x)$$



Properties of Legendre polynomials:

(1) $P_n(x=1) = 1$ (2) $P_n(x=-1) = (-1)^n$

(3) $P_n(x)$ is an even function of x for even n and is an odd function of x for odd n.

(4) $P_n(-x) = (-1)^n P_n(x)$ (5) $P_n'(x=1) = \frac{1}{2}n(n+1)$

(6) $P_n(x=0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}\dfrac{1.3.5.....(n-1)}{2.4.6.....n} & \text{if } n \text{ is even} \end{cases}$

Generating function of Legendre polynomial:

$$\left(1-2xz+z^2\right)^{-1/2}=\sum_{n=0}^{\infty}z^n P_n(x)$$

Co-efficient of z^n in the exapansion of the generating function is the Legendre polynomial of order n.

Orthonormal property of Legendre polynomial:

$$\int_{-1}^{+1}P_m(x)P_n(x)dx=\frac{2}{2n+1}\delta_{mn}$$

Recurrence relations for Legendre polynomials:

(i) $n P_n=(2n-1)xP_{n-1}-(n-1)P_{n-2}$ (ii) $n P_n=xP_n'-P_{n-1}'$

(iii) $(2n+1)P_n=P_{n+1}'-P_{n-1}'$ (iv) $(n+1)P_n=P_{n+1}'-xP_n'$

(v) $(1-x^2)P_n'=n(P_{n-1}-xP_n)$ (vi) $(1-x^2)P_n'=(n+1)(xP_n-P_{n+1})$

(B) Bessel Differential Equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-n^2\right)y=0$$

Note: $x=0$ is a regular singular point of the Bessel differential equation.

Solution:

Bessel function order n:

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

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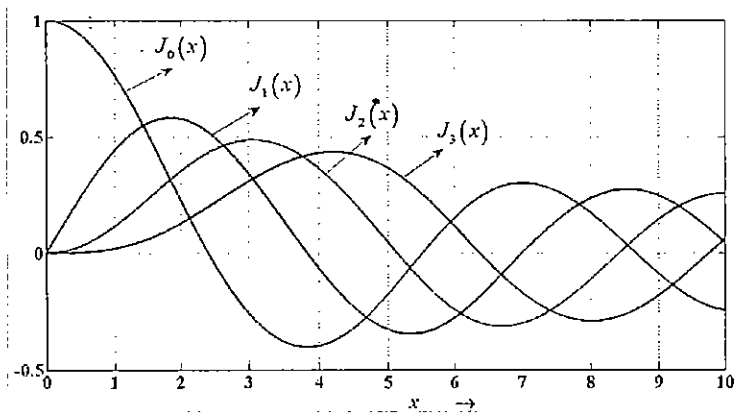
First few order of Bessel functions are

$$J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot4^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots..;\ J_1(x)=\frac{x}{2}-\frac{x^3}{2^2 4}+\frac{x^5}{2^2\cdot4^2\cdot6}-\ldots..\text{ and so on}$$



Properties of Bessel functions:

(i) $J_n(x)$ are oscillatory function with varying period and decreasing amplitude.

(ii) $J_n(x)$ is an even function for even n and $J_n(x)$ is an odd function for odd n.

(iii) $J_0(x)$ has nodes at $x = 2.4048, 5.5201$ ..... and $J_1(x)$ has nodes at $x = 3.8317, 7.015$ ..... and so on.

(iv) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

Note:

(1) Since the Bessel differential equation involves only n^2, then

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r}$$

is also a solution of the Bessel differential equation.

(2) If n is not an integer, then $J_n(x)$ and $J_{-n}(x)$ are linearly independent solutions.

(3) If n is an integer, then $J_n(x)$ and $J_{-n}(x)$ are linearly dependent and are related by

$$J_{-n}(x) = (-1)^n J_n(x)$$

Generating function of Bessel function:

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

The coefficient of z^n in the expansion of the generating function is the Bessel function of order n.

Orthonormal property of Bessel Fucntion:

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{\delta_{\alpha\beta}}{2}\left[J_{n+1}(\alpha)\right]^2$$

Recurrence relations for Bessel Fucntion:

(i) $x J_n'(x) = nJ_n - xJ_{n+1}$ (ii) $x J_n' = -nJ_n x J_{n-1}$

(iii) $2J_n' = J_{n-1} - J_{n+1}$ (iv) $2n J_n = x\left[J_{n-1} + J_{n+1}\right]$

(v) $\frac{d}{dx}\left[x^{-n}J_n\right] = -x^{-n}J_{n+1}$ (vi) $\frac{d}{dx}\left[x^n J_n\right] = x^n J_{n-1}$

(C) Hermite Differential Equation

$$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2ny=0$$

Note: $x=0$ is an ordinary point of the Hermite differential equation

Solution:

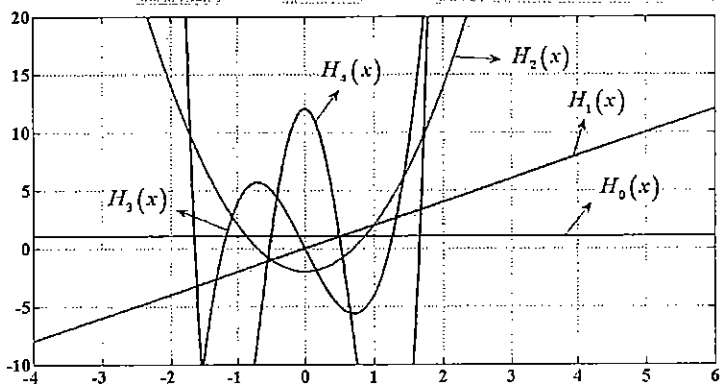
Hermite polynomial of order n:

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

First few orders of Hermite polynomials are:

$$H_0(x)=1 \qquad H_3(x)=8x^3-12x$$
$$H_1(x)=2x \qquad H_4(x)=16x^4-48x^2+12$$
$$H_2(x)=4x^2-2$$

Note: $H_n(x)$ will be even function if n is even and will be odd function if n is odd.



Generating function for Hermite polynomials:

$$e^{2zx-z^2}=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}z^n$$

Coefficient of z^n in the expansion of e^{2zx-z^2} is $\dfrac{H_n(x)}{n!}$

Orthonormal property of Hermite polynomials:

$$\int_{-\infty}^{\infty}e^{-x^2}H_n(x)H_m(x)\,dx=2^n n!\sqrt{\pi}\,\delta_{nm}$$

Recurrence relation for Hermite polynomials:

(i) $H_n'(x) = 2n\,H_{n-1}(x)$ (ii) $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$

(iii) $H_n'(x) = 2xH_n(x) - H_{n+1}(x)$ (iv) $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$

(v) $H_n(-x) = (-1)^n H_n(x)$

(D) Laguerre Differential Equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0$$

Note: $x = 0$ is a regular singular point of the Laguerre differential equation.

Solution:

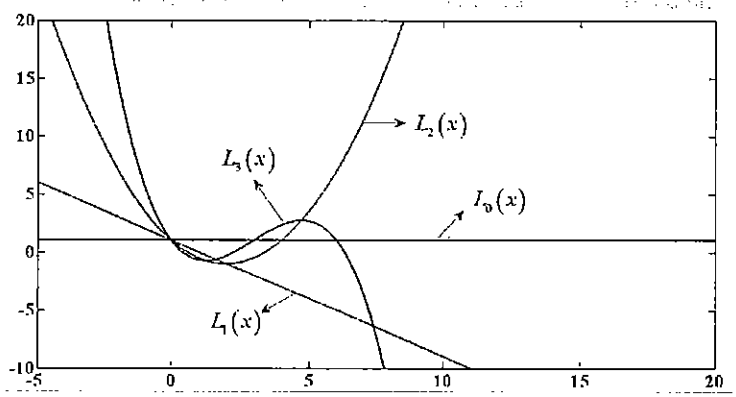
Laguerre polynomial of order n: $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right)$

First few orders of Laguerre polynomials are

$$L_0(x) = 1 \quad L_1(x) = 1 - x \quad L_2(x) = \frac{1}{2}\left(x^2 - 4x + 2\right)$$

$$L_3(x) = \frac{1}{6}\left(-x^3 + 9x^2 - 18x + 6\right)$$

Note: Laguerre polynomials are neither even nor odd functions of x.



Generating function for Laguerre polynomials:

$$\frac{e^{-xz/(1-z)}}{(1-z)} = \sum_{n=0}^{\infty} z^n L_n(x)$$

Orthogonal property of Laguerre polynomials:

$$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\,dx = \delta_{nm}$$

Question 1. The differential equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$ has the complete solution, in terms of arbitrary constants A and B as **[TIFR 2013]**

(a) $A\exp x + Bx\exp x$

(b) $A\exp x + B\exp(-x)$

(c) $A\exp x + Bx\exp(-x)$

(d) $x\{A\exp x + B\exp(-x)\}$

Soln: $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$

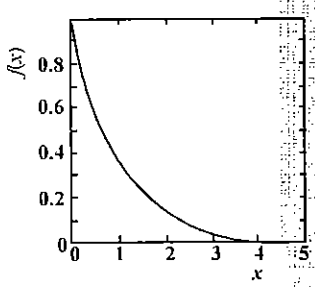
Auxillary equation : $m^2-2m+1=0 \Rightarrow (m-1)^2=0 \Rightarrow m=1,1$

Hence, $y(x)=(A+Bx)e^x$

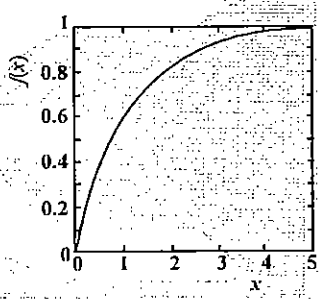
Correct option is (a)

Question 2. The solution of the integral equation $f(x)=x-\int_0^x dt\, f(t)$ has the graphical form **[TIFR 2014]**

(a)



(b)



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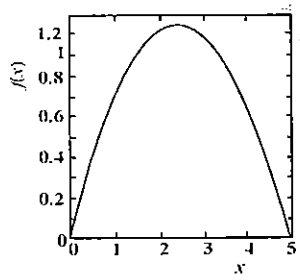
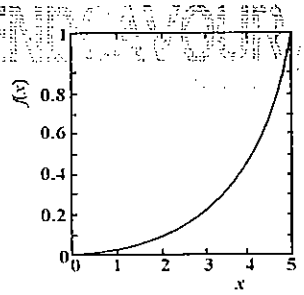


(d)



Soln: $f(x)=x-\int_0^x dt\, f(t) \Rightarrow f'(x)=1-f(x) \Rightarrow f'(x)+f(x)=1 \Rightarrow \frac{df(x)}{dx}+f(x)=1.$

Therefore, I.F. $=e^{\int 1dx}=e^x$

Therefore, solution will be $e^x f(x)=\int 1\times e^x\,dx+C \Rightarrow f(x)=\frac{e^x+C}{e^x} \Rightarrow f(x)=1+Ce^{-x}$

Hence, as $x\to\infty$; $f(x)\to 1$. Here, only option (b) is satisfying this condition.

Correct option is (b)

Question 3. Consider the differential equation $\frac{d^2y}{dx^2} = -4\left(y + \frac{dy}{dx}\right)$ with the boundary conditions that $y(x) = 0$ at $x = \frac{1}{5}$. When plotted as a function of x, for $x \geq 0$, we can say with certainity that the value of y

(a) first increases, then decreases to zero. **[TIFR 2015]**

(b) first decreases, then increases to zero.

(c) has an extremum in the range $0 < x < 1$.

(d) oscillates from positive to neagtive with amplitude decreasing to zero.

Soln: $\frac{d^2y}{dx^2} = -4\left(y + \frac{dy}{dx}\right) \Rightarrow \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

Auxillary equation: $m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2$

Hence, $y(x) = (c_1 + c_2 x)e^{-2x}$.

Also, $y(x) = 0$ at $x = \frac{1}{5}$

Therefore, $0 = \left(c_1 + c_2 \times \frac{1}{5}\right)e^{-2/5} \Rightarrow c_1 + \frac{c_2}{5} = 0 \Rightarrow c_1 = -\frac{c_2}{5}$

Therefore, $y(x) = c_2\left(\frac{1}{5} - x\right)e^{-2x}$

Now for maxima and minima $y'(x) = c_2(-1)e^{-2x} + c_2\left(\frac{1}{5} - x\right) \times -2e^{-2x} = 0$

$\Rightarrow c_2\left[-1 - \frac{2}{5} + 2x\right]e^{-2x} = 0 \quad \Rightarrow 2x = \frac{7}{5} \Rightarrow x = \frac{7}{10}$

Hence, $y(x)$ has an extremum (either maxima or minima) at the point 7/10. We can not claim it to have maxima or minima at $x = \frac{7}{10}$, becuase we don't know the sign of c_2.

Correct option is (c)

Question 4. The function $y(x)$ satisfies the differential equation $x\frac{dy}{dx} = y(\ln y - \ln x + 1)$ with the initial condition $y(1) = 3$. What will be the value of $y(3)$? **[TIFR 2016]**

Soln: We have, $x\frac{dy}{dx} = y(\ln y - \ln x + 1) \Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right)\left[\ln\left(\frac{y}{x}\right) + 1\right]$

Let, $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \quad v + x\frac{dv}{dx} = v[\ln v + 1] \Rightarrow \quad v + x\frac{dv}{dx} = v\ln v + v \quad \Rightarrow \quad \int\frac{dv}{v\ln v} = \int\frac{dx}{x}$

$\Rightarrow \quad \ln(\ln v) = \ln x + c \quad \Rightarrow \quad \ln\left(\ln\frac{y}{x}\right) = \ln x + c \quad$ (puting $v = \frac{y}{x}$)

Given: $y(1)=3 \Rightarrow \quad \ln(\ln(3))=c$

$\therefore \quad \ln\left[\ln\left(\frac{y}{x}\right)\right]=\ln x+\ln(\ln 3) \Rightarrow \quad \ln\left(\frac{y}{x}\right)=x\cdot\ln 3 \Rightarrow y=xe^{(x\ln 3)}$

$\therefore \quad y(3)=3\times e^{(3\ln 3)}=3\times 27=81$

Correct answer is (81)

Question 5. Write down $x(t)$, where $x(t)$ is the solution of the following differential equation

$$\left(\frac{d}{dt}+2\right)\left(\frac{d}{dt}+1\right)x=1,$$

with the boundary conditions $\left.\frac{dx}{dt}\right|_{t=0}=0, \quad x(t)|_{t=0}=-\frac{1}{2}.$ **[TIFR 2017]**

Soln: $\left(\frac{d}{dt}+2\right)\left(\frac{d}{dt}+1\right)x=1 \Rightarrow \frac{d^2x}{dt^2}+\frac{dx}{dt}+2\frac{dx}{dt}+2x=1 \Rightarrow \frac{d^2x}{dt^2}+3\frac{dx}{dt}+2x=1$

Auxillary equation : $m^2+3m+2=0 \Rightarrow (m+1)(m+2)=0 \Rightarrow m=-1,-2$

Therefore, C.F. $=c_1e^{-x}+c_2e^{-2x}$

P.I. $=\frac{1}{D^2+3D+2}\times 1\,e^0=\frac{1}{2}$ (Put $D=0$)

Therefore, $x(t)=c_1e^{-x}+c_2e^{-2x}+\frac{1}{2}$

Now, $x(t)|_{t=0}=c_1+c_2+\frac{1}{2}=-\frac{1}{2} \Rightarrow c_1+c_2=1 \quad \ldots(1)$

And $\frac{dx(t)}{dt}=-1c_1e^{-x}-2c_2e^{-2x}$

$\left.\frac{dx(t)}{dt}\right|_{t=0}=-c_1-2c_2=0 \Rightarrow c_1=-2c_2 \quad \ldots(2)$

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From Eq. (1) and Eq. (2) ; $c_2=-1$ and $c_1=2$.

Therefore, $x(t)=-e^{-x}+2e^{-2x}+\frac{1}{2}$.

Question 6. If $y(x)$ satisfies the differential equation $y''-4y'+4y=0$ with boundry conditions $y(0)=1, y'(0)=0$, then $y\left(-\frac{1}{2}\right)=$ **[TIFR 2018]**

(a) $\frac{2}{e}$ (b) $\frac{1}{2}\left(e+\frac{1}{e}\right)$ (c) $\frac{1}{e}$ (d) $-\frac{e}{2}$

Soln: Auxillary equation : $m^2-4m+4=0 \Rightarrow (m-2)^2=0 \Rightarrow m=2,2$

Therefore, $y=(c_1+c_2x)e^{2x}$

Given $y(0)=1 \Rightarrow (c_1+c_2.0)e^{2.0}=1 \Rightarrow c_1=1$

Here, $y'(x)=c_2e^{2x}+(c_1+c_2x)(2e^{2x})$

Given $y'(0)=0 \Rightarrow c_2.e^{2.0}+(c_1+c_2.0)(2.e^{2.0})=0 \Rightarrow c_2+2c_1=0 \Rightarrow c_2=-2$

Therefore, the solution will be $y(x)=(1-2x)e^{2x}$

So, $y\left(x=-\frac{1}{2}\right)=\left[1-2.\left(-\frac{1}{2}\right)\right]e^{2\left(-\frac{1}{2}\right)}=\frac{2}{e}$

Correct option is (a)

JEST PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. Consider the differential equation $\frac{dG(x)}{dx}+kG(x)=\delta(x)$

where k is a constant. Which of the following statement is true? **[JEST 2013, 2015]**

(a) Both G(x) and G'(x) are continuous at $x=0$

(b) G(x) is continuous at $x=0$ but G'(x) is not.

(c) G(x) is discontinuous at $x=0$

(d) The continuity properties of G(x) and G'(x) at $x=0$ depend on the value of k.

Soln: $\frac{dG(x)}{dx}+kG(x)=\delta(x)$

I.F. $=e^{\int k\,dx}=e^{kx}$

Therefore, the required solution will be

$$G(x)e^{kx}=\int\delta(x)e^{kx}+c$$

Continuity check at x = 0:

L.H.L. $=\lim_{x\to0^-}G(x)e^{kx}=\lim_{x\to0^-}\int\delta(x)e^{kx}+c=c$

R.H.L. $=\lim_{x\to0^+}G(x)e^{kx}=\lim_{x\to0^+}\int\delta(x)e^{kx}+c=c$

At $x=0$, $G(0)e^{kx}\big|_{x=0}=e^0+c=1+c$

Since, L.H.L. = R.H.L. $\neq G(0)$. Hence, $G(x)$ is not continuous at $x=0$.

Correct option is (c)

Question 2. What are the solutions to $f''(x)-2f'(x)+f(x)=0$? **[JEST 2014]**

(a) c_1e^x/x (b) c_1x+c_2/x (c) $c_1xe^x+c_2$ (d) $c_1e^x+c_2xe^x$

Soln: $f''(x)-2f'(x)+f(x)=0$

Auxillary equation $m^2-2m+1=0 \Rightarrow (m-1)^2=0 \Rightarrow m=1,1$

Therefore, $f(x)=(c_1+c_2x)e^x$

Correct option is (d)

JNU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The general solution of the differential equation, $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$ in terms of two arbitrary constants A and B, is **[JNU 2012]**

(a) $e^{1/x}\left(A\cos\left(\frac{1}{x}\right)+B\sin\left(\frac{1}{x}\right)\right)$ (b) $Ax+\frac{B}{x}$

(c) $Ax + Bxe^x$ (d) $Ax + Bx^2$

Soln: Given equation is Cauchy Euler equation of second order.

$x=e^z \Rightarrow \log x=z \Rightarrow \frac{1}{x}=\frac{dz}{dx}$

Putting $x\frac{dy}{dx}=\frac{dy}{dz}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$, we get $\Rightarrow \frac{d^2y}{dz^2}-3\frac{dy}{dz}+2y=0$

Auxilliary equation: $m^2-3m+2=0 \Rightarrow (m-2)(m-1)=0 \Rightarrow m=1,2$

Therefore, solution will be $y=c_1e^z+c_2e^{2z} \Rightarrow y=c_1x+c_2x^2$.

Correct option is (d)

Question 2. For $x > 0$, the solution of the differential equation

$$\frac{dy}{dx}+2xy-\frac{1}{x}e^{-x^2}=0$$

(with the condition that y = 0 at x = 1) is (In the following: $Erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\zeta^2}d\zeta$) **[JNU 2015]**

(a) $e^{-x^2}\ln x$ (b) $\ln x\, Erf(x)$ (c) $(x-1)e^{-x^2}$ (d) $1-\frac{1}{x}$

Soln: $\frac{dy}{dx}+2xy-\frac{1}{x}e^{-x^2}=0$

I.F. $=e^{\int 2x\,dx}=e^{x^2}$

Therefore, the required solution will be $y(x)e^{x^2}=\int\frac{1}{x}e^{-x^2}e^{x^2}+c$

$\Rightarrow y(x)e^{x^2}=\ln x+c \Rightarrow y(x)=e^{-x^2}\ln x+ce^{-x^2}$

Also, $y=0$ at $x=1 \Rightarrow 0=0+ce^{-1} \Rightarrow \boxed{c=0}$

Therefore, $y(x)=\ln e^{-x^2}$.

Correct option is (a)

Question 3. Consider the following differential equation $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$. It is given that $\frac{d^2y}{dx^2}=1, \frac{dy}{dx}=0$ and $y=1$ at $x=0$. Then the value of y at $x=2$ is nearest to **[JNU 2016]**

(a) 30 (b) 42 (c) 21 (d) 63 (d) 12

Soln: $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$

Auxilliary equation is $\left(D^3-3D+2D\right)y=0$

For non-trivial solution $D^3-3D+2D=0 \Rightarrow D\left(D^2-3D+2\right)=0$

Hence, $\boxed{D=0,1,2}$

$\therefore \quad y=C_1e^{0x}+C_2e^{x}+C_3e^{2x}$

$y=C_1+C_2e^{x}+C_3e^{2x}$ ... (1)

$\frac{dy}{dx}=C_2e^{x}+2C_3e^{2x}$ ... (2)

And $\frac{d^2y}{dx^2}=C_2e^{x}+4C_3e^{2x}$ ... (3)

$\because \quad y=1$ at $x=0$, from (1), $1=C_1+C_2+C_3$ ... (4)

$\frac{dy}{dx}=0$ at $x=0$, from (2), $0=C_2+2C_3$ ... (5)

And $\frac{d^2y}{dx^2}=1$ at $x=0$, from (3), $1=C_2+4C_3$ ... (6)

Solving equations (4), (5) and (6), we get

$$\boxed{C_1=\frac{3}{2}, C_2=-1, C_3=\frac{1}{2}}$$

Hence, $y(x)=\frac{3}{2}-e^{x}+\frac{1}{2}e^{2x}$

Therefore, at $x=2$,

$$y(x=2)=\frac{3}{2}-e^{2}+\frac{1}{2}e^{4}\simeq 21$$

Correct option is (c)

Question 4. The general solution of the ordinary differential equation, $\frac{d^2y}{dx^2}+y+1=0$ is (A and B being constants). **[JNU 2017]**

(a) x^2+Ax+B (b) $A(x-1)^3+B$ (c) $A\sin x$

(d) $A\sin(x+B)$ (e) $A\cos x+B\sin x-1$

Soln: $\frac{d^2y}{dx^2}+y+1=0$ Let $z=y+1$ $\therefore$ $\frac{d^2y}{dx^2}=\frac{d^2z}{dx^2}$

$\therefore$ $\frac{d^2z}{dx^2}+z=0$

Auxillary equation $m^2+1=0 \Rightarrow m=\pm i$

Therefore, the solution will be $z=A\cos x+B\sin x$

$\therefore$ $y=A\cos x+B\sin x-1$

Correct option is (d)

DU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The general solution of the differential equation $\frac{d^3y}{dx^3}+\frac{1}{x}\frac{d^2y}{dx^2}+\frac{1}{x^2}\frac{dy}{dx}=0$ can be expressed as (C_1, C_2 and C_3 are constants): **[DU 2014]**

(a) $C_1 + C_2x^{1+i} + C_3x^{1-i}$ (b) $C_1 + C_2x + C_3x^i$

(c) $C_1x + C_2x^2 + C_3x^3$ (d) $C_1x + C_2x^{1+i} + C_3x^{1-i}$

Soln: $\frac{d^3y}{dx^3}+\frac{1}{x}\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=0 \Rightarrow x^3\frac{d^3y}{dx^3}+x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$

Given equation is Cauchy Euler equation of third order $x=e^z \Rightarrow \log x=z \Rightarrow \frac{1}{x}=\frac{dz}{dx}$

Putting $x\frac{dy}{dx}=\frac{dy}{dz}$, $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$ and $x^3\frac{d^3y}{dx^3}=\frac{d^3y}{dz^3}-3\frac{d^2y}{dz^2}+2\frac{dy}{dz}$,

we get $\Rightarrow \frac{d^3y}{dz^3}-2\frac{d^2y}{dz^2}+2\frac{dy}{dz}=0$

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Auxilliary equation: $m^3-2m^2+2m=0 \Rightarrow m\left(m^2-2m+2\right)=0 \Rightarrow m=0,1+i,1-i$

Therefore, the solution will be $y=c_1e^{0z}+c_2e^{(1+i)z}+c_3e^{(1-i)z} \Rightarrow y=c_1+c_2x^{1+i}+c_3x^{1-i}$

Correct option is (a)

Question 2. The most general solution of the differential equation $\frac{d^2y}{dx^2}+\frac{4dy}{dx}+4y=0$ is (here 'a' and 'b' are constants)

[DU 2016]

(a) $ae^{-2x}+be^{2x}$ (b) $ae^{-2x}+be^{-2x}$ (c) $ae^{-2x}+bxe^{-2x}$ (d) $ae^{2x}+bxe^{2x}$

Soln: Auxilliary equation: $m^2+4m+4=0 \Rightarrow (m+2)^2=0 \Rightarrow m=-2,-2$

Hence, $y=(a+bx)e^{-2x}$.

Correct option is (c)

Question 3. The solution to the differential equation $\frac{dy}{dx}+2xy=x$ given that $y(x=0)=0$ is

[DU 2017]

(a) $y=(1-e^{-x^3})$ (b) $y=\frac{1}{4}(1-e^{-x^2})$ (c) $y=\frac{1}{2}(1-e^{-x^2})$ (d) $y=\frac{1}{2}(1-e^{-x})$

Soln: I.F. $=e^{\int 2x\,dx}=e^{x^2}$

Therefore, the solution will be $y\times e^{x^2}=\int x e^{x^2}dx+c$

Putting $x^2=t\Rightarrow 2x\,dx=dt$, we get

$$ye^{x^2}=\int\frac{dt}{2}e^t+c=\frac{1}{2}e^t+c \quad\Rightarrow ye^{x^2}=\frac{1}{2}e^{x^2}+c\Rightarrow y=\frac{1}{2}+ce^{-x^2}$$

Since, $y(0)=0$, put $x=0$, $y=0$ $\Rightarrow 0=\frac{1}{2}+c\Rightarrow c=-\frac{1}{2}$

Therefore, $y=\frac{1}{2}\left(1-e^{-x^2}\right)$

Correct option is (c)

HCU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The most general solution of the equation $\frac{d^2x}{dt^2}+\omega^2x=ce^{-\alpha t}$ can be written as

(a) $A\sin\omega t+B\cos\omega t$ (b) $A\sin\omega t+B\cos\omega t+Ce^{-\alpha t}$ **[H.C.U.-2010]**

(c) $A\sin\omega t+B\cos\omega t+C(\alpha^2+\omega^2)e^{-\alpha t}$ (d) $A\sin\omega t+B\cos\omega t+Ce^{-\alpha t}/(\alpha^2+\omega^2)$

Soln: $\frac{d^2x}{dt^2}+\omega^2x=ce^{-\alpha t}$

Auxilliary equation: $m^2+\omega^2=0\Rightarrow m=\pm i\,\omega$

Therefore, C.F. $=A\cos\omega t+B\sin\omega t$

$$\text{P.I.}=\frac{1}{D^2+\omega^2}ce^{-\alpha t}=\frac{ce^{-\alpha t}}{\alpha^2+\omega^2}$$

Therefore, solution will be $x=A\cos\omega t+B\sin\omega t+\frac{ce^{-\alpha t}}{\alpha^2+\omega^2}$

Correct option is (a)

Question 2. The solution of the differential equation $\frac{dy}{dx}=x+xy$ is to be obtained under the condition that $y=0$ at $x=0$. The solution is given by **[H.C.U.-2011]**

(a) $y=\frac{x^2}{2}$ (b) $y=e^{-x^2/2}-1$ (c) $y=e^{x^2/2}-1$ (d) $y=\frac{e^x}{2}-1$

Soln: $\frac{dy}{dx}=x+xy\quad\Rightarrow\int\frac{dy}{1+y}=\int x\,dx\quad\Rightarrow\ln(1+y)=\frac{x^2}{2}+c$

$\because \quad y = 0$ at $x = 0 \quad \Rightarrow \ln(1) = 0 + c \Rightarrow c = 0$

Therefore, the solution will be, $\ln(1+y) = \dfrac{x^2}{2} \Rightarrow y = e^{x^2/2} - 1$

Correct option is (c)

Question 3. Complete solution of the differential equation

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}$$

is given by (where c_1 and c_2 are arbitrary constants) **[H.C.U.-2012]**

(a) $c_1e^x + c_2e^{3x} - e^{2x}$ (b) $c_1e^x + c_2e^{6x} - e^{2x}$ (c) $c_1e^x + c_2e^{6x} - \frac{1}{4}e^{2x}$ (d) $c_1e^x + c_2e^{3x} - \frac{1}{4}e^{2x}$

Soln: Auxilliary equation: $m^2 - 7m + 6 = 0 \Rightarrow m = 1, 6$

$\text{C.F.} = c_1 e^x + c_2 e^{6x}$

$$\text{P.I.} = \frac{1}{D^2 - 7D + 6} \times e^{2x} = \frac{e^{2x}}{4 - 14 + 6} = \frac{e^{2x}}{-4}$$

$\therefore \quad y = c_1 e^x + c_2 e^{6x} - \frac{1}{4} e^{2x}$

Correct option is (c)

Question 4. Two independent solutions of the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

may be chosen to be

(a) $e^{-2x} + e^{-x}, 2\left(e^{-2x} - e^{-x}\right)$ (b) $e^{-2x} + e^{-x}, 2\left(e^{-2x} + e^{-x}\right)$

(c) $e^{-2x} + e^{-x}, x\left(e^{-2x} - e^{-x}\right)$ (d) e^{-2x}, e^{x} **[H.C.U.-2012]**

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Soln: Auxilliary equation: $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$

$\therefore \quad y = c_1 e^{-x} + c_2 e^{-2x}$ where c_1 and c_2 are arbitary constants.

Correct option is (a)

Question 5. The solution to the differential equation $\dfrac{d^2x}{dt^2} = t^2$ is given by **[H.C.U.-2015]**

(a) $x = \dfrac{t^3}{3} + A$ (b) $x = \dfrac{t^3}{3} + At + B$ (c) $x = \dfrac{t^4}{12} + At + B$ (d) $x = \dfrac{t^4}{3} + \dfrac{t^2}{2} + A$

where A and B are constants.

Soln: $\dfrac{d^2x}{dt^2} = t^2 \quad \Rightarrow x = \dfrac{t^4}{12} + At + B$

Correct option is (c)

Question 6. The general soution to the differential equation $\frac{d^2y}{dx}-\frac{2dy}{dx}+2y=0$, is

(a) $e^x\left[A\cos x+iB\sin x\right]$ (b) $e^{ix}\left[A\cos x+iB\sin x\right]$

(c) $e^{ix}\left[A\cos x-iB\sin x\right]$ (d) $e^x\left[A\cos x-iB\sin x\right]$ **[H.C.U.-2015]**

where A and B are arbitrary constants.

Soln: Auxilliary equation: $m^2-2m+2=0 \Rightarrow m=1+i,1-i$

$\therefore \quad y=e^x\,(A\cos x+B\sin x)$

Correct option is ( )

Question 7. On changing variable from x to t where $t=\log x$, the differential equation $x^2\frac{d^2y}{dx^2}+\alpha x\frac{dy}{dx}+\beta y=0$ becomes

(a) $e^{2t}\frac{d^2y}{dx^2}+\alpha e^t\frac{dy}{dx}+\beta y=0$ (b) $\frac{d^2y}{dx^2}+\alpha\frac{dy}{dx}+\beta y=0$

(c) $\frac{d^2y}{dx^2}+\alpha\frac{dy}{dx}+(\alpha+\beta)y=0$ (d) $\frac{d^2y}{dx^2}+(\alpha-1)\frac{dy}{dx}+\beta y=0$ **[H.C.U.-2015]**

Soln: $t=\log x$ or $x=e^t$; $\frac{dx}{dt}=e^t$

If $\frac{dy}{dx}=D$ and $\frac{dy}{dt}=D_1$, then $x^2\frac{d^2y}{dx^2}=D_1(D_1-1)y$ and $x\frac{dy}{dx}=D_1y$

Therefore, given differential equation $x^2\frac{d^2y}{dx^2}+\alpha x\frac{dy}{dx}+\beta y=0$

is now $D_1(D_1-1)y+\alpha D_1y+\beta y=0$

$\Rightarrow D_1^2y+(\alpha-1)D_1+\beta y=0 \quad \Rightarrow \frac{d^2y}{dt^2}+(\alpha-1)\frac{dy}{dt}+\beta y=0$

Correct option is (d)

Question 8. For $\lambda>0$ and $\lambda\in\mathbb{R}$, what are the eigenfunctions of the following differential equation $y''+\lambda y=0$, for the boundary conditions $y(0)=0,\ y'(\pi)=0$? **[H.C.U.-2017]**

(a) $C_1\cos\left(\frac{2n-1}{2}x\right)$ (b) $C_2\sin\left(\frac{2n-1}{2}x\right)$

(c) $C_1\cos(nx)+C_2\sin(nx)$ (d) $C_1\cos(2n-1)x+C_2\sin(2n-1)x$

Soln: Auxilliary equation: $m^2+\lambda=0 \Rightarrow m=+i\sqrt{\lambda},-i\sqrt{\lambda}$

Therefore, C.F. $=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$

Now given condition $y(0)=0 \Rightarrow \boxed{A=0}$

Therefore, $y = B\sin\sqrt{\lambda}x$

Here, $\frac{dy}{dx} = B\cos\sqrt{\lambda}x \times \sqrt{\lambda}$

Put $\frac{dy}{dx}=0$ and $x=\pi \Rightarrow \cos\sqrt{\lambda}\pi = 0 \Rightarrow \sqrt{\lambda}\pi = \left(\frac{2n-1}{2}\right)\pi \Rightarrow \sqrt{\lambda} = \left(\frac{2n-1}{2}\right)$

$\therefore \quad y = A\sin\left(\frac{2n-1}{2}x\right)$

Correct option is (a)

BHU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. Consider the solution $y(t)$ of the equation **[B.H.U-2010]**

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+k^2y=\exp(-kt)$$

where k is a constant and whose value and derivative with respect to t are zero at $t=0$. Then $y(t)$ is

(a) $\frac{1}{2}\exp(-kt)$ (b) $\frac{1}{2}t\exp(-kt)$ (c) $\frac{1}{2}t^2\exp(-kt)$ (d) $\frac{1}{2}\exp(-kt^2)$

Soln: $\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+k^2y=\exp(-kt)$

Auxillary equation, $(D^2+2kD+k^2)y=0$

For non-trivial solution, $D^2+2kD+k^2=0$

$(D+k)^2=0 \Rightarrow D=-k,-k$

Therefore, C.F. $=(c_1+c_2t)e^{-kt}$

P.I. $=\frac{1}{D^2+2kD+k^2}e^{-kt} = t\times\frac{1}{2D+2k}e^{-kt} = t^2\times\frac{1}{2}e^{-kt}$

Hence, $y(t)=(c_1+c_2t)e^{-kt}+\frac{t^2e^{-kt}}{2}$

Since, $y(t=0)=0$. Hence, $0=c_1$

$y'(t)=c_2te^{-kt}\times -k + c_2e^{-kt} + \frac{t^2}{2}e^{-kt}\times -k + \frac{2te^{-kt}}{2}$

Since, $y'(t=0)=0 \Rightarrow 0=c_2$

Hence, $y(t)=\frac{t^2}{2}e^{-kt}$.

Correct option is (c)

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Question 2. The generating function for Legendre polynomial $P_n(x)$ is **[B.H.U-2011]**

(a) $\left(1-2xt+t^2\right)^{-1/2}$ (b) $\left(1+2xt-t^2\right)^{-1/2}$ (c) $\left(1-2xt+t^2\right)^{1/2}$ (d) $\left(1+2xt-t^2\right)^{1/2}$

Soln: Correct option is (a)

Question 3. The value of $\frac{d}{dx}\left[x^n J_n(x)\right]$ is **[B.H.U-2011]**

(a) $x^n J_{n+1}(x)$ (b) $-x^{-n} J_{n+1}(x)$ (c) $x^n J_{n-1}(x)$ (d) $x^{n-1} J_{n-1}(x)$

Soln: Correct option is (b)

Question 4. If the Rodrigue's formula for a polynomial is **[B.H.U-2011]**

$$(-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

then the polynomial is

(a) Legendre (b) Hermite (c) Lagure (d) Bessel

Soln: Correct option is (b)

Question 5. $a_1 = 1$ and $a_n = n(1+a_{n-1})$ for $n = 2,3$. Define $P_n = \prod_{k=1}^{n}\left(1+\frac{1}{a_k}\right)$. Then $\lim_{n\to\infty} P_n$ is

(a) 1 (b) ∞ (c) e (d) $1+e$ **[B.H.U-2012]**

Soln: Correct option is (a)

Question 6. The differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$ has solution as **[B.H.U-2013]**

(a) Bessel function $J_n(x)$ (b) Hermite polynomial $H_n(x)$

(c) Legendre polynomial $P_n(x)$ (d) Laguerre polynomial $L_n(x)$

Soln: Correct option is (a)

CAREER ENDEAVOUR

Question 7. The spherical Bessel function $j_n(x)$ is related to the Bessel function $J_n(x)$ by the relation

(a) $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$ (b) $j_n(x) = \sqrt{\frac{\pi}{2x}} J_n(x)$ **[B.H.U-2013]**

(c) $j_n(x) = \sqrt{\frac{2x}{\pi}} J_{n+\frac{1}{2}}(x)$ (d) $j_n(x) = \sqrt{\frac{2x}{\pi}} J_n(x)$

Soln: Correct option is (a)

Question 8. The value of $J_0(x)$ at $x=0$ is given by **[B.H.U-2013]**

(a) 1 (b) 0 (c) ∞ (d) $\sqrt{\pi}$

Soln: $J_0(x)$ at $x=0$ is equal to 1.

Correct option is (a)

Question 9. The Legendre series expansion for the function $f(\theta)=\sin^2\left(\frac{\theta}{2}\right)$ can be given as :

(a) $\frac{1}{2}[P_0(\cos\theta)-P_1(\cos\theta)]$ (b) $\frac{1}{2}[P_2(\cos\theta)-P_3(\cos\theta)]$ **[B.H.U-2014]**

(c) $[P_1(\cos\theta)-P_2(\cos\theta)]$ (d) $[P_0(\cos\theta)+P_1(\cos\theta)]$

Soln: Since, $P_0(x)=1$; $P_1(x)=x$; $P_2(x)=\frac{1}{2}(3x^2-1)$

Now, $f(\theta)=\sin^2\left(\frac{\theta}{2}\right)=\frac{1-\cos\theta}{2}$

Put $x=\cos\theta$, $f(\theta)=\frac{1}{2}[P_0(\cos\theta)-P_1(\cos\theta)]$.

Correct option is (a)

Question 10. The generating function for Hermite polynomial is $g(t,x)=e^{-t^2+2tx}$. The zero of the Hermie polynomial $H_2(x)$ on the positive x-axis is **[B.H.U-2014]**

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{4}$ (d) $\frac{\sqrt{3}}{2}$

Soln: We know, Hermite polynomial $H_2(x)=4x^2-2$

$H_2(x)=0 \Rightarrow 4x^2-2=0$

$x^2=\frac{1}{2} \Rightarrow x=\pm\frac{1}{\sqrt{2}}$

So, on positive x-axis, $x=\frac{1}{\sqrt{2}}$

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Correct option is (b)

Question 11. The function $e^{\frac{x(t^2-1)}{2t}}$ is a generating function of **[B.H.U-2015]**

(a) Hermite polynomials (b) Legendre polynomials

(c) Laguerre polynomials (d) Bessel functions

Soln: **Correct option is (d)**

Question 12. The value of $J_{1/2}(x)$ is **[B.H.U-2015]**

(a) $\sqrt{\frac{2}{\pi x}}\cos x$ (b) $\sqrt{\frac{2}{\pi x}}\sin x$ (c) $\sqrt{\frac{\pi x}{2}}\cos x$ (d) $\sqrt{\frac{\pi x}{2}}\sin x$

Soln: $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x.$

Correct option is (b)

Question 13. Which of the following is true ?

(a) $J_{5/2}(x)=\frac{3}{x}J_{3/2}(x)-J_{1/2}(x)$ (b) $J_{5/2}(x)=\frac{x}{3}J_{3/2}(x)-J_{1/2}(x)$ **[B.H.U-2015]**

(c) $J_{5/2}(x)=\frac{3}{x}J_{3/2}(x)+J_{1/2}(x)$ (d) $J_{5/2}(x)=\frac{x}{3}J_{3/2}(x)+J_{1/2}(x)$

Soln: Using the recurrence relation, $2nJ_n=x\left[J_{n+1}(x)+J_{n-1}(x)\right]$

Put $n=\frac{3}{2}$, $2\times\frac{3}{2}J_{3/2}(x)=x\left[J_{5/2}(x)+J_{1/2}(x)\right]$

$J_{5/2}(x)=\frac{3}{x}J_{3/2}(x)-J_{1/2}(x)$.

Correct option is (a)

Question 14. Legendre polynomial $P_n(x)$ for order $n=0$ has, for any value of x, value **[B.H.U-2015**

(a) 0 (b) ∞ (c) 1 (d) none of these

Soln: For $n=0$, $P_0(x)=1$

Correct option is (c)

Question 15. For $P_n(x)$, which of the following is true ? **[B.H.U-2015]**

(a) $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2+1)^n$ (b) $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(1-x^2)^n$

(c) $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ (d) $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x-1)^n$

Soln: **Correct option is (c)**

Question 16. The value of the Legendre polynomial $P_n(x)$ for $n=3$, is **[B.H.U-2016]**

(a) $\left(\frac{5x^3+3x}{2}\right)$ (b) $\left(\frac{5x^3+3x}{1}\right)$ (c) $\left(\frac{5x^3-3x}{2}\right)$ (d) $\left(\frac{5x^3-3x}{1}\right)$

Soln: $P_3(x)=\frac{1}{2}(5x^3-3x)$.

Correct option is (c)

Question 17. One of the following recurrence relations is correct for the Hermite polynomial $H_n(x)$:

(a) $H_n'(x)=2(n+1)H_{n+1}(x)$ (b) $H_n'(x)=2(n-1)H_{n-1}(x)$

(c) $H_n'(x)=2nH_{n+1}(x)$ (d) $H_n'(x)=2nH_{n-1}(x)$ **[B.H.U-2016]**

Soln: **Correct option is (d)**

Question 18. The value of the Bessal functions $[J_{1/2}(x)]^2+[J_{-1/2}(x)]^2$ is one of the following:

(a) $\frac{2}{\pi x}$ (b) $\frac{\pi x}{2}$ (c) $\sqrt{\frac{2}{\pi x}}$ (d) $\sqrt{\frac{\pi x}{2}}$ **[B.H.U-2016]**

Soln: We have, $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$; $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$

Hence, $\left[J_{1/2}(x)\right]^2+\left[J_{-1/2}(x)\right]^2=\frac{2}{\pi x}$.

Correct option is (a)

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Question 19. The generating functional for the Bessel function is :

(a) $e^{x(z+z^{-1})} = \sum_{n=-\infty}^{+\infty} J_n(x) z^n$ (b) $e^{x(z-z^{-1})} = \sum_{n=-\infty}^{+\infty} J_n(x) z^n$

(c) $e^{\frac{x(z+z^{-1})}{2}} = \sum_{n=-\infty}^{+\infty} J_n(x) z^n$ (d) $e^{\frac{x(z-z^{-1})}{2}} = \sum_{n=-\infty}^{+\infty} J_n(x) z^n$ **[B.H.U-2016]**

Soln: **Correct option is (d)**

Question 20. One of the following is true :

(a) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^{n-1} J_{n-1}(x)$ (b) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^{n+1} J_{n-1}(x)$

(c) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n+1}(x)$ (d) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$ **[B.H.U-2016]**

Soln: **Correct option is (d)**

Question 21. In the Bessel equation $x^2 \frac{d^2y}{dx^2} + (x^2 - n^2) y + x \frac{dy}{dx} = 0$ **[B.H.U-2016]**

(a) $x = 1$ is a regular singular point (b) $x = 1$ is an irregular singular point

(c) $x = 0$ is a regular singular point (d) $x = 0$ is an irregular singular point

Soln: $x = 0$ is a regular singular point.

Correct option is (c)

Question 22. The Wronskian $W(y_1, y_2)$ for the two solutions $y_1(x)$ and $y_2(x)$ of the second order differential equation is

(a) $W = y_1 y_2' - y_2 y_1' \left(y_1' = \frac{dy_1}{dx} \right)$ (b) $W = y_1 y_2' + y_2 y_1' \left(y_2' = \frac{dy_2}{dx} \right)$ **[B.H.U-2016]**

(c) $W = y_1 y_2' - y_1' y_2'$ (d) $W = y_1 y_2' + y_1' y_2'$

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Soln: We have, $W(y_1\ y_2\ x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$

Correct option is (a)

Question 23. The coefficient of t^n in the expansion of the function $e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}$ is called **[B.H.U-2017]**

(a) The Legendre function (b) The Bessel function of first kind of order n

(c) Laugurre function (d) Hermite polynomial of order n

Soln: **Correct option is (b)**

Question 24. $H_{n-1}(x) + H_{n+1}(x)$ equals (where terms have their usual meaning) **[B.H.U-2017]**

(a) $\frac{2n}{x} H_n(x)$ (b) $2n\, H_n(x)$ (c) $2H_n'(x)$ (d) $H_{n+2}(x)$

Soln: **None of the options are matching.**

Chapter 4

FOURIER SERIES

4.1 Periodic Function

If the value of function $f(x)$ repeats itself after equal interval of x, then $f(x)$ is said to be periodic function of x.

$$f(x) = f(x+T) = f(x+2T) = ...$$

then $f(x)$ is said to be periodic function of x with period T.

Example: (i) $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$: $T = 2\pi$

(ii) $\tan x$, $\cot x$: $T = \pi$

(iii) $\sin(5x + 2\pi) = \sin 5x = \sin 5\left(x + \frac{2\pi}{5}\right)$: $T = \frac{2\pi}{5}$

4.2 Fourier Series Expansion

A non-sinusoidal periodic function can be expressed by a sum of a set of sinusoidal oscillating functions or periodic functions i.e. mainly sine and cosine functions (or complex exponentials). This series is known as 'Fourier Series' expansion of the non-sinisoidal periodic function.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

a_0, a_n, b_n are constants, known as Fourier co-efficients.

A periodic function $f(x)$ can be expanded into fourier series which consists of the following terms:

(i) a_0 : a constant term or D. C. term

(ii) $a_1 \cos\frac{\pi x}{L}$ and $b_1 \sin\frac{\pi x}{L}$: Fundamental frequency term i.e term having lowest frequency

(iii) $a_2 \cos\frac{2\pi x}{L}, a_3 \cos\frac{3\pi x}{L}$ ....., $b_2 \sin\frac{2\pi x}{L}, b_3 \sin\frac{3\pi x}{L}$ ..... : Overtones i.e terms having frequencies integer multiple of fundamental frequency term.

Note: When a function and its derivative are continuous, then the function can be expanded by powers of x by Taylor series. But for both continuous and discontinuous functions, fourier series expansion is possible under certain conditions.

Dirichlet's conditions:

Suppose a function $f(x)$ is defined in the interval (a,b) and if

(i) it is single-valued in the interval (a,b)

(ii) it is bounded in the interval (a,b)

(iii) it has at most a finite number of maxima and minima in the interval (a,b)

(iv) it has only a finite number of discontinuity in the interval (a,b)

(v) $f(x+T)=f(x)$ where $[T=b-a]$, for all values of x outside $[a,b]$ then,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$$

Useful Integrals

(1) $\int_a^b \sin\frac{n\pi x}{L}dx=\int_a^b \cos\frac{n\pi x}{L}dx=0$

(2) $\int_a^b \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx=\int_a^b \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx=L\,\delta_{mn}$

(3) $\int_a^b \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx=0$ (4) $\sin n\pi=0,\ \cos n\pi=(-1)^n$

Note: In all of these integrals, $(b-a)$ should be $2L$.

Evaluation of Fourier co-efficients:

CASE 1:

Suppose a function $f(x)$ is defined in the interval $(-L,L)$, then the Fourier series expansion of $f(x)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$$

where $a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx$; $a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx$; $b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx$

CASE 2:

Suppose a function $f(x)$ is defined in the interval $(0,2L)$, then the Fourier series expansion of $f(x)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$$

where $a_0=\frac{1}{L}\int_{0}^{2L}f(x)dx$; $a_n=\frac{1}{L}\int_{0}^{2L}f(x)\cos\frac{n\pi x}{L}dx$; $b_n=\frac{1}{L}\int_{0}^{2L}f(x)\sin\frac{n\pi x}{L}dx$

CASE 3:

Suppose a function $f(t)$ is defined in the interval $\left(0, \frac{2\pi}{\omega}\right)$, then the Fourier series expansion of $f(x)$ is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$a_0 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)dt \; ; \; a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\cos n\omega t \, dt \; ; \; b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin n\omega t \, dt$$

CASE 4:

Suppose a function $f(t)$ is defined in the interval $\left(-\frac{\pi}{\omega}, \frac{\pi}{\omega}\right)$, then the Fourier series expansion of $f(x)$ is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$a_0 = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)dt \; ; \; a_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\cos n\omega t \, dt \; ; \; b_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\sin n\omega t \, dt$$

Nature of the fourier series for even function and odd function:

CASE 1: EVEN FUNCTION

A function is said to be even if $f(-x) = f(x)$ i.e. nature of the function is symmetric about $x = 0$.

We know that

$$\int_{-L}^{L} F(x)dx = \begin{cases} 2\int_0^L F(x)dx & \text{if } F(x) \text{ is an even function} \\ 0 & \text{if } F(x) \text{ is an odd function} \end{cases}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = 0$$

This series will contain only cosine terms and the fourier series expansion can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

CASE 2: ODD FUNCTION

A function is said to be odd if $f(-x) = -f(x)$ i.e the nature of the function is anti-symmetric about $x = 0$.

We know that

$$\int_{-L}^{L} F(x)\,dx = \begin{cases} 2\int_{0}^{L} F(x)\,dx & \text{if } F(x) \text{ is an even function} \\ 0 & \text{if } F(x) \text{ is an odd function} \end{cases}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

This series will contain only sine terms and the fourier series expansion can be written as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Example 1: Find the fourier series representing, $f(x) = x,\ 0 < x < 2\pi$ and sketch its graph from $x = -4\pi$ to $x = 4\pi$.

Soln: $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$

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$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

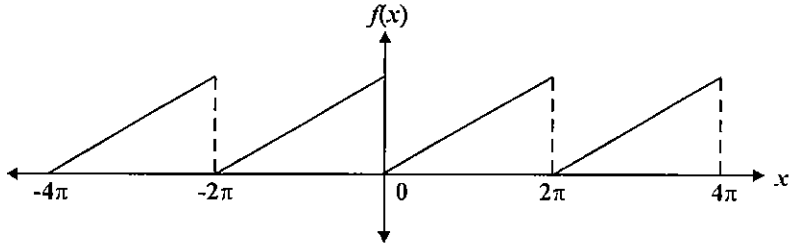
$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi\cos 2n\pi}{n}\right] = -\frac{2}{\pi}$$

The required fourier series expansion of the function:

$$f(x) = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\ldots\right]$$



Example 2: Given that $f(x)=x+x^2$ for $-\pi<x<\pi$, find the Fourier expression of $f(x)$. Deduce that $\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.....$

Soln: $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$

$$=\frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right]=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n}-(2x+1)\frac{(-\cos nx)}{n^2}+(2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2}-(-2\pi+1)\frac{\cos(-n\pi)}{n^2}\right]=\frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right]=\frac{4(-1)^n}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$=\frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right)-(2x+1)\left(\frac{-\sin nx}{n^2}\right)+2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos n\pi}{n}+2\frac{\cos n\pi}{n^3}+(-\pi+\pi^2)\frac{\cos n\pi}{n}-2\frac{\cos n\pi}{n^3}\right]$$

$$=\frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right]=-\frac{2}{n}(-1)^n$$

The required fourier series series expansion of the function:

$$x+x^2=\frac{\pi^3}{3}+4\left[-\cos x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+....\right]-2\left[-\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+....\right]$$

Putting $x=\pi$ we get, $\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\right]$

Putting $x=-\pi$ we get, $-\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.....\right]$

$$\Rightarrow\frac{4\pi^2}{3}=8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.....\right]\Rightarrow\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.....=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Example 3: Find the fourier series to represent the function $f(x)$ given by

$$f(x)=\begin{cases}-k & \text{for } -\pi<x<0\\ k & \text{for } 0<x<\pi\end{cases}$$

hence show that $1-\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots=\frac{\pi}{4}$

Soln: Given function is odd in nature, so $a_0=0$ and $a_n=0$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}-k\sin nx\,dx+\int_{0}^{\pi}k\sin nx\,dx\right]$$

$$=\frac{1}{\pi}k\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{0}-\left\{\frac{\cos nx}{n}\right\}_{0}^{\pi}\right]$$

$$=\frac{1}{\pi}k\left[\frac{1}{n}-\frac{(-1)^n}{n}-\frac{(-1)^n}{n}+\frac{1}{n}\right]=\frac{1}{\pi}k\left[\frac{2}{n}-\frac{2(-1)^n}{n}\right]$$

If n is even $b_n=0$

If n is odd $b_n=\frac{4k}{n\pi}$

Thus required fourier sine series is

$$f(x)=\frac{4k}{\pi}\sin x+\frac{4k}{3\pi}\sin 3x+\frac{4k}{5\pi}\sin 5x+\ldots\ldots$$

$$\Rightarrow\quad f(x)=\frac{4k}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\ldots.\right]$$

Putting $x=\frac{\pi}{2}$, we get

$$k=\frac{4k}{\pi}\left[\sin\frac{\pi}{2}+\frac{1}{3}\sin\frac{3\pi}{2}+\frac{1}{5}\sin\frac{5\pi}{2}+\ldots.\right]$$

$$\Rightarrow 1=\frac{4}{\pi}\left[1+\frac{1}{3}(-1)+\frac{1}{5}(1)+\frac{1}{7}(-1)+\ldots\ldots\right]=\frac{4}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots.\right]\Rightarrow\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots$$

Example 4: Find the fourier series of the function:

$$f(x)=\begin{cases}-1 & \text{for } -\pi<x<-\frac{\pi}{2}\\ 0 & \text{for } -\frac{\pi}{2}<x<\frac{\pi}{2}\\ +1 & \text{for } \frac{\pi}{2}<x<\pi\end{cases}$$

Soln: $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\,dx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}0\,dx+\frac{1}{\pi}\int_{\pi/2}^{\pi}1\,dx$

$$=\frac{1}{\pi}[-x]_{-\pi}^{-\pi/2}+\frac{1}{\pi}[x]_{\pi/2}^{\pi}=\frac{1}{\pi}\left[\frac{\pi}{2}-\pi+\pi-\frac{\pi}{2}\right]=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx \quad =\frac{1}{\pi}\int_{-\pi}^{\pi/2}(-1)\cos nxdx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\cos nxdx+\frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\cos nxdx$$

$$=-\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{-\pi/2}+\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi} \quad =-\frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right]+\frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right]=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

$$=\frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\sin nxdx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\sin nxdx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(1)\sin nxdx$$

$$=\frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2}-\frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$=\frac{1}{n\pi}\left[\cos\frac{n\pi}{2}-\cos n\pi\right]-\frac{1}{n\pi}\left(\cos n\pi-\cos\frac{n\pi}{2}\right)=\frac{2}{n\pi}\left[\cos\frac{n\pi}{2}-\cos n\pi\right]$$

$b_1=\frac{2}{\pi}$, $b_2=-\frac{2}{\pi}$, $b_3=\frac{2}{3\pi}$, therefore the fourier series expansion becomes

$$f(x)=\frac{1}{\pi}\left[2\sin x-2\sin 2x+\frac{2}{3}\sin 3x+.....\right]$$

Example 5: Find the fourier seires of the function defined as

$$f(x)=\begin{cases}x+\pi, & \text{for } 0\le x\le\pi\\ -x-\pi, & \text{for } -\pi\le x<0\end{cases} \text{ and } f(x+2\pi)=f(x)$$

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Soln: $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}f(x)dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)dx$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)dx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)dx=\frac{1}{\pi}\left(-\frac{x^2}{2}-\pi x\right)_{-\pi}^{0}+\frac{1}{\pi}\left(\frac{x^2}{2}+\pi x\right)_{0}^{\pi}$$

$$=\frac{1}{\pi}\left(\frac{\pi^2}{2}-\pi^2\right)+\frac{1}{\pi}\left(\frac{\pi^2}{2}+\pi^2\right)=\pi\left(\frac{1}{2}-1\right)+\pi\left(\frac{1}{2}+1\right)=\pi$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\cos nxdx+\frac{1}{\pi}\int_{0}^{\pi}f(x)\cos nxdx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\cos nxdx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\cos nxdx$$

$$=\frac{1}{\pi}\left[(-x-\pi)\frac{\sin nx}{n}-(-1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[(x+\pi)\frac{\sin nx}{n}-(-1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{0}^{\pi}$$

$$=\frac{1}{\pi}\left[-\frac{1}{n^2}+\frac{(-1)^n}{n^2}\right]+\frac{1}{\pi}\left[\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right]=\frac{2}{n^2\pi}\left[(-1)^n-1\right]$$

$=\frac{-4}{n^2\pi}$, if n is odd

$= 0$, if n is even

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\sin nxdx+\frac{1}{\pi}\int_{0}^{\pi}f(x)\sin nxdx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\sin nxdx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\sin nxdx$$

$$=\frac{1}{\pi}\left[(-x-\pi)\left(\frac{-\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}+\frac{1}{\pi}\left[(x+\pi)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{\pi}{n}\right]+\frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n+\frac{\pi}{n}\right]=\frac{1}{n}\left[(1)-2(-1)^n\right]=\frac{2}{n}\left[1-(-1)^n\right]$$

$=\frac{4}{n}$, if n is odd

$= 0$, if n is even

The fourier series expansion of the function is

$$f(x)=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+\ldots+b_1\sin x+b_2\sin 2x+\ldots.$$

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\ldots.\right)+4\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\ldots.\right)$$

Example 6: Find the Fourier series expansion of the periodic function of period 2π

$$f(x)=x^2, \quad -\pi\le x\le\pi$$

Hence, find the sum of the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$

Soln: $f(x)=x^2, \quad -\pi\le x\le\pi$

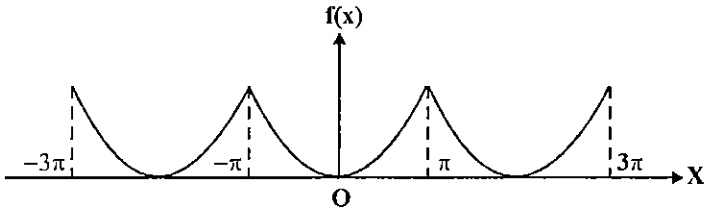
This is an even function, $\therefore\ b_n=0$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}x^2dx=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}=\frac{2\pi^2}{3}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx=\frac{2}{\pi}\int_0^{\pi}x^2\cos nxdx$$

$$=\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-(2x)\left(-\frac{\cos nx}{n^2}\right)+(2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{\pi^2\sin n\pi}{n}+\frac{2\pi\cos n\pi}{n^2}-\frac{2\sin n\pi}{n^3}\right]=\frac{4(-1)^n}{n^2}$$



Fourier series expansion: $x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right]$

On putting $x = 0$, we have $0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots\right]$

$$\Rightarrow \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots = \frac{\pi^2}{12}$$

Example 7: Find the Fourier Series expansion for the function

$$f(x) = x\cos x, -\pi < x < \pi$$

Soln: Since $x\cos x$ is an odd function therefore, $a_0 = a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{\pi} + \frac{1}{\pi}\left[-x\frac{\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{(n+1)} - \frac{\cos(n-1)x}{(n-1)}\right\} + 1.\left\{\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi.\left\{-\frac{\cos(n+1)\pi}{(n+1)} - \frac{\cos(n-1)\pi}{(n-1)}\right\}\right] \Rightarrow b_n = \left\{-\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1}\right\}, \; n \neq 1$$

$$b_n = -(-1)^{n+1}\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= -\left\{\frac{1}{(n+1)} + \frac{1}{(n-1)}\right\} = \frac{-2n}{n^2-1} \qquad \text{If n is odd, } n \neq 1$$

But $b_n = \left\{\frac{1}{n+1} + \frac{1}{n-1}\right\} = \frac{2n}{n^2-1}$, If n is even, $n \neq 1$

If n = 1, then $b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x.\sin x\,dx = \frac{1}{\pi}\int_0^{\pi} x.\sin 2x\,dx$

$$= \frac{1}{\pi}\left[x.\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[\pi\left(-\frac{1}{2}\right)\right] = -\frac{1}{2}$$

$$\therefore \quad x\cos x = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots\ldots = -\frac{1}{2}\sin x + \frac{4\sin 2x}{2-1} - \frac{6\sin 3x}{3^2-1} + \ldots\ldots$$

Example 8: Represent the following function by fourier sine series

$$f(t) = \begin{cases} t, & 0 < t \le \dfrac{\pi}{2} \\ \dfrac{\pi}{2}, & \dfrac{\pi}{2} < t \le \pi \end{cases}$$

Soln: $b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\int_0^{\pi/2} t\sin nt\,dt + \frac{2}{\pi}\int_{\pi/2}^{\pi}\frac{\pi}{2}\sin nt\,dt$

$$= \frac{2}{\pi}\left[t\left(-\frac{\cos nt}{n}\right) - (1)\left(-\frac{\sin nt}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\frac{\pi}{2}\left[-\frac{\cos nt}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \left[-\frac{\cos n\pi}{n} + \frac{\cos\frac{n\pi}{2}}{n}\right]$$

$$b_1 = \frac{2}{\pi}\left[-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right] + \left[-\cos\pi + \cos\frac{\pi}{2}\right] = \frac{2}{\pi}[0+1] + [1] = \frac{2}{\pi} + 1$$

$$b_2 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\pi}{2} + \frac{\sin\pi}{2}\right] + \left[\frac{\cos 2\pi}{2} + \frac{\cos\pi}{2}\right] = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{(-1)}{2} + 0\right] + \left[-\frac{1}{2} - \frac{1}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\right] - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$b_3 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{3\pi}{2}}{3} + \frac{\sin\frac{3\pi}{2}}{3^2}\right] + \left[-\frac{\cos 3\pi}{3} + \frac{\cos\frac{3\pi}{2}}{3}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}(0) - \frac{1}{9}\right] + \left[\frac{1}{3} + 0\right] = -\frac{2}{9\pi} + \frac{1}{3}$$

$$f(t) = \left(\frac{2}{\pi} + 1\right)\sin t - \frac{1}{2}\sin 2t + \left(-\frac{2}{9\pi} + \frac{1}{3}\right)\sin 3t + \ldots\ldots$$

Example 9: A periodic function of period 4 is defined as

$$f(x) = |x|. \quad -2 < x < 2$$

Find its Fourier series expansion.

Soln: $f(x) = |x|. \quad -2 < x < 2$

$$\Rightarrow \qquad f(x) = \begin{cases} x, & 0 < x < 2 \\ -x, & -2 < x < 0 \end{cases}$$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\int_0^2 x\,dx + \frac{1}{2}\int_{-2}^{0}(-x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 + \frac{1}{2}\left[\frac{-x^2}{2}\right]_{-2}^{0} = \frac{1}{4}(4-0) + \frac{1}{4}(0+4) = 2$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx = \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2 + \frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (-1)\left(-\frac{4}{n^2\pi^2}\right)\cos\frac{n\pi x}{2}\right]_{-2}^{0}$$

$$= \frac{1}{2}\left[0 + \frac{4}{n^2\pi^2}(-1)^n - \frac{4}{n^2\pi^2}\right] + \frac{1}{2}\left[0 - \frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2}(-1)^n\right]$$

$$= \frac{1}{2}\frac{4}{n^2\pi^2}\left[(-1)^n - 1 - 1 + (-1)^n\right] = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= -\frac{8}{n^2\pi^2} \qquad \text{(If n is odd)}$$

$$= 0 \qquad \text{(if n is even)}$$

$b_n = 0$ as $f(x)$ is even function

Therefore, the fourier series exapansion will be

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{2} + a_2\cos\frac{2\pi x}{2} + \ldots + b_1\sin\frac{\pi x}{2} + b_2\sin\frac{2\pi x}{2} + \ldots$$

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2} + \ldots\right]$$

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Example 10: The fourier series of the function $f(x) = 1 - |x|,\ -\pi < x < \pi$ is given as

(a) $\left(1 - \frac{\pi}{2}\right) + \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$

(b) $\left(1 - \frac{\pi}{2}\right) + \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \cdots\right)$

(c) $\left(1 - \frac{\pi}{2}\right) + \frac{4}{\pi}\left(\cos x + \frac{\sin 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\sin 7x}{7^2} + \cdots\right)$

(d) None of the above

Soln: As we know, Fourier series of $f(x)$ in the interval (a, b) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{2\pi nx}{b-a}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{2\pi nx}{b-a}\right)$$

Here, $f(x) = 1 - |x|,\ -\pi < x < \pi \Rightarrow f(x) = \begin{cases} 1+x & ;\ -\pi < x < 0 \\ 1-x & ;\ 0 < x < \pi \end{cases}$

Clearly, $f(-x)=f(x)$, Hence $f(x)$ is an even function.

Therefore, $b_n=0$.

Now, $a_0=\frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$=\frac{2}{\pi}\int_0^{\pi}(1-x)\,dx=\frac{2}{\pi}\left(x-\frac{x^2}{2}\right)_0^{\pi}=2\left(1-\frac{\pi}{2}\right)$$

$$a_n=\frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{2\pi nx}{2\pi}dx=\frac{1}{\pi}2\int_0^{\pi}(1-x)\cos(nx)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\cos nx\,dx-\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx=\frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}-\frac{2}{\pi}\left\{x\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right\}_0^{\pi}$$

$$=0-\frac{2}{\pi}\left\{\frac{\cos n\pi}{n^2}-\frac{1}{n^2}\right\}=-\frac{2}{\pi}\left[\frac{(-1)^n-1}{n^2}\right]$$

$$\therefore\quad a_n=\begin{cases}\dfrac{4}{\pi n^2}, & \text{if } n=\text{odd}\\ 0, & \text{if } n=\text{even}\end{cases}$$

$\therefore$ Fourier series of $f(x)$ is

$$f(x)=\left(1-\frac{\pi}{2}\right)+\sum_{n=1,3,5}^{\infty}\frac{4}{\pi n^2}\cos nx=\left(1-\frac{\pi}{2}\right)+\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right)$$

Correct option is (a)

Example 11: If $f(x)=\begin{cases}-k & ;\ -\pi<x<0\\ k & ;\ 0\le x\le\pi\end{cases}$, and a_0, a_n, b_n are the fourier co-efficients of fourier expansion of $f(x)$, then which of the following statement is CORRECT?

(a) $a_0=0,\ a_n=0,\ b_n\neq 0$ (b) $a_0\neq 0,\ a_n=0,\ b_n=0$

(c) $a_0\neq 0,\ a_n\neq 0,\ b_n=0$ (d) $a_0=0,\ a_n\neq 0,\ b_n=0$

Soln: Since, $f(x)$ is an odd function.

Hence, $a_0=0,\ a_n=0,\ b_n\neq 0$.

Correct option is (a)

Example 12: If the fourier expansion of $f(x)=\begin{cases}\pi+x & ;\ -\pi<x<0\\ 0 & ;\ 0\le x\le\pi\end{cases}$ is given by

$$f(x)=\frac{\pi}{4}+\frac{2}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\cdots\right]-\left[\frac{\sin x}{1}+\frac{\sin 2x}{2}+\cdots\right],$$

then the value of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ is equal to

(a) $-\frac{\pi}{8}$ (b) $\frac{\pi}{8}$ (c) $\frac{\pi^2}{8}$ (d) $\frac{3\pi^2}{8}$

Soln: $f(x)=\frac{\pi}{4}+\frac{2}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\cdots\right]-\left[\frac{\sin x}{1}+\frac{\sin 2x}{2}+\cdots\right]$,

Putting $x=0$ on both sides, we get

$\Rightarrow \frac{1}{2}\left(\lim_{x\to 0^+} f(x)+\lim_{x\to 0^-} f(x)\right)=\frac{\pi}{4}+\frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right]-0$ (since, $f(x)$ is discontinuous at $x=0$)

$\Rightarrow \frac{1}{2}(0+\pi)=\frac{\pi}{4}+\frac{\pi}{2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right]$

Therefore, $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$.

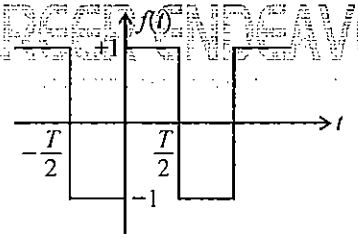
Correct option is (c)

Example 13: The fourier series of the square wave function, as shown in figure is

(a) $\frac{1}{\pi}(\sin\omega t+\sin 2\omega t+\cdots)$ (b) $\frac{1}{\pi}\left(\sin\omega t+\frac{\sin 2\omega t}{2}+\frac{\sin 3\omega t}{3}+\cdots\right)$

(c) $\frac{4}{\pi}\left(\sin\omega t+\frac{\sin 3\omega t}{3}+\frac{\sin 5\omega t}{5}+\cdots\right)$ (d) $\frac{4}{\pi}\left(1+\frac{\sin 2\omega t}{2}+\frac{\sin 4\omega t}{4}+\cdots\right)$

$\left(\text{Here, } \omega=\frac{2\pi}{T} \text{ is the angular frequency}\right)$



Soln: From the figure, $f(t)=\begin{cases}1 & ; \quad 0<t<\frac{T}{2}\\ -1 & ; \quad -\frac{T}{2}<t<0\end{cases}$

Clearly, $f(t)$ is an odd function. (anti-symmetric about $x=0$)

Hence, $a_0=a_n=0$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

$$=\frac{2}{T}\times 2\int_0^{T/2}(+1)\sin\left(\frac{2\pi nt}{T}\right)dt=\frac{4}{T}\left(\frac{-\cos\left(\frac{2\pi nt}{T}\right)}{\frac{2\pi n}{T}}\right)_0^{T/2}=\frac{4}{T}\times\frac{T}{2\pi n}(-\cos n\pi+1)$$

Therefore, $b_n = \begin{cases} \frac{4}{2\pi n} \times 2 & \text{if} \quad n = \text{odd} \\ 0 & \text{if} \quad n = \text{even} \end{cases}$

Hence, fourier series $f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$

$$f(t) = \frac{4}{\pi}\left[\frac{\sin\left(\frac{2\pi t}{T}\right)}{2} + \frac{\sin\left(\frac{6\pi t}{T}\right)}{3} + \cdots\right]$$

$$f(t) = \frac{4}{\pi}\left[\frac{\sin \omega t}{2} + \frac{\sin 3\omega t}{3} + \cdots\right]$$

Correct option is (c)

Example 14: The fourier series for $|\sin\theta|$ in the range $-\pi \le \theta \le \pi$ is given by

(a) $\frac{1}{\pi} - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2m\theta}{4m^2 - 1}$

(b) $\frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2m\theta}{4m^2 - 1}$

(c) $-\frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2m\theta}{4m^2 - 1}$

(d) None of these

Soln: We have $f(\theta) = |\sin\theta|, \ -\pi \le x \le \pi$

$$f(\theta) = \begin{cases} -\sin\theta & ; \quad -\pi \le \theta < 0 \\ \sin\theta & ; \quad 0 \le \theta \le \pi \end{cases}$$

Since, $f(\theta)$ is an even function, hence $b_n = 0$.

Now, $a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta = \frac{1}{\pi} \times 2\int_0^{\pi} \sin\theta\, d\theta = \frac{4}{\pi}$

and $a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cos\left(\frac{2\pi n\theta}{2\pi}\right) d\theta = \frac{1}{\pi} \times 2\int_0^{\pi} \sin\theta \cos n\theta\, d\theta$

$$a_n = \frac{2}{\pi}\left[\frac{\sin\theta \sin n\theta}{n} + \frac{\cos\theta \cos n\theta}{n^2}\right]_0^{\pi} \frac{n^2}{n^2 - 1}$$

$$= \frac{2}{\pi}\frac{n^2}{n^2 - 1}\left[\frac{-\cos\pi - 1}{n^2}\right] = -\frac{2}{\pi}\frac{1}{n^2 - 1}\left((-1)^n + 1\right)$$

$$a_n = \begin{cases} -\frac{4}{\pi} \times \frac{1}{n^2 - 1}, & \text{if} \quad n = \text{even} \\ 0, & \text{if} \quad n = \text{odd} \end{cases}$$

Hence, fourier series,

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$$f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{2\pi n\theta}{2\pi}\right)=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=2,4,6}^{\infty}\frac{\cos(n\theta)}{n^2-1}$$

Put $n=2m,\ m=1,2,3,\ldots$

$$\therefore\ f(\theta)=\frac{2}{\pi}-\frac{4}{\pi}\sum_{m=1,2,3,\ldots}^{\infty}\frac{\cos(2m\theta)}{4m^2-1}$$

Correct option is (b)

Example 15: If the fourier series of $f(x)$ in the interval $[-l, l]$, where

$$f(x)=\begin{cases}L/4 & ;\ -L\le x\le -\frac{L}{2}\\ x^2/L & ;\ -\frac{L}{2}\le x\le\frac{L}{2}\\ L/4 & ;\ \frac{L}{2}\le x\le L\end{cases}$$

is given by $f(x)=\frac{L}{6}+\frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}A_n\cos\frac{n\pi x}{2}$, then A_n is equal to

(a) $\cos\left(\frac{n\pi}{2}\right)-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$ (b) $\sin\left(\frac{n\pi}{2}\right)-2\cos\left(\frac{n\pi}{2}\right)$

(c) $1-\frac{2}{n\pi}\cos\frac{n\pi}{2}$ (d) $1-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$

Soln: Since, $f(-x)=f(x)$. Hence, $f(x)$ is an even function. Hence, $b_n=0$

$$a_0=\frac{2}{2L}\int_{-L}^{L}f(x)\,dx=\frac{1}{L}2\int_0^L f(x)\,dx=\frac{2}{L}\int_0^{L/2}f(x)\,dx+\frac{2}{L}\int_{L/2}^{L}f(x)\,dx=\frac{2}{L}\int_0^{L/2}\frac{x^2}{L}dx+\int_{L/2}^{L}\frac{L}{4}dx=\frac{L}{3}$$

$$a_n=\frac{2}{2L}\int_{-L}^{L}f(x)\cos\left(\frac{2\pi nx}{2L}\right)dx=\frac{1}{L}2\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$=\frac{2}{L}\int_0^{L/2}\frac{x^2}{L}\cos\left(\frac{n\pi x}{L}\right)dx+\frac{2}{L}\int_{L/2}^{L}\frac{L}{4}\cos\left(\frac{n\pi x}{L}\right)dx$$

$$a_n=\frac{2}{L}\left[\frac{x^2}{L}\cdot\frac{\sin(n\pi x/L)}{n\pi/L}-\frac{2}{n\pi}\left\{x\times-\frac{\cos(n\pi x/L)}{n\pi/L}-\frac{\sin(n\pi x/L)}{(n\pi/L)^2}\right\}\right]_0^{L/2}+\frac{2}{L}\times\frac{L}{4}\left(\frac{\sin(n\pi x/L)}{n\pi/L}\right)_{L/2}^{L}$$

$$=\frac{2}{L}\left[\left\{\frac{L^2}{4}\times\frac{\sin(n\pi/2)}{n\pi}-\frac{2}{n\pi}\left(-\frac{L}{2}\times\frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right)-\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)\right\}-0\right]+\frac{1}{2}\times\frac{L}{n\pi}\left[-\sin\left(\frac{n\pi}{2}\right)\right]$$

$$=\frac{2L}{\pi^2n^2}\left[\cos\left(\frac{n\pi}{2}\right)-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

Hence, on comparing with given fourier series, we have

$$A_n=\cos\left(\frac{n\pi}{2}\right)-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right).$$

Correct option is (a)

Example 16: If $f(x)=\begin{cases}\cos x & ; \quad 0\le x\le \pi \\ -\cos x & ; \quad -\pi \le x\le 0\end{cases}$, then which of the following statements is/are CORRECT about the fourier expansion of $f(x)$ in the interval $[-\pi, \pi]$.

(a) Ratio of third harmonic to fourth harmonic is zero.

(b) Ratio of third harmonic to fourth harmonic is $32/15\pi$.

(c) $f(x)=\frac{4}{\pi}\left[\frac{2}{1\cdot 3}\sin 2x+\frac{4}{3\cdot 5}\sin 4x+\cdots\right]$, and $f(x)$ converges to zero at $x=0$.

(d) $f(x)=\frac{4}{\pi}\left[\frac{2}{1^2\cdot 3^2}\sin 2x+\frac{4}{3^2\cdot 5^2}\sin 4x+\cdots\right]$, and $f(x)$ converges to zero at $x=0$.

Soln: Since, $f(-x)=-f(x)$, hence $f(x)$ is an odd function. Therefore, $a_0=a_n=0$.

$$b_n=\frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{2n\pi x}{2\pi}\right)dx=\frac{2}{\pi}\int_0^{\pi}\cos x\sin(n\pi)\,dx$$

$$=\frac{2}{\pi}\times\left(\frac{n^2}{n^2-1}\right)\left(-\frac{\cos x\cos nx}{n}-\frac{\sin x\sin nx}{n^2}\right)_0^{\pi}=\frac{2}{\pi}\left(\frac{n^2}{n^2-1}\right)\left(\frac{\cos n\pi}{n}+\frac{1}{n}\right)$$

$$b_n=\frac{2}{\pi}\left(\frac{n}{n^2-1}\right)\left((-1)^n+1\right)\Rightarrow b_n=\begin{cases}\dfrac{4n}{\pi(n^2-1)}, & \text{if} \quad n=\text{even} \\ 0, & \text{if} \quad n=\text{odd}\end{cases}$$

Hence, $b_1=b_3=b_5\cdots=0$

Therefore, ratio of third to fourth harmonic $=\frac{b_3}{b_4}=0$.

Also, fourier series, $f(x)=\sum_{n=1}^{\infty}b_n\sin(nx)=\frac{4}{\pi}\left(\frac{2}{3}\sin 2x+\frac{4}{15}\sin 4x+\frac{6}{35}\sin 6x+\cdots\right)$

$$=\frac{4}{\pi}\left(\frac{2}{1\cdot 3}\sin 2x+\frac{4}{3\cdot 5}\sin 4x+\frac{6}{5\cdot 7}\sin 6x+\cdots\right)$$

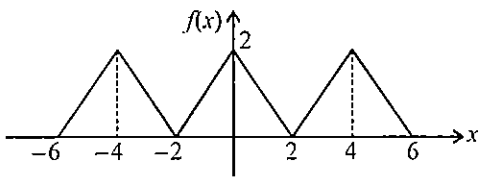
and at $x=0$, $f(x)$ converges to $\frac{1}{2}\left(\lim_{x\to 0^-}f(x)+\lim_{x\to 0^+}f(x)\right)=\frac{1}{2}(-1+1)=0$.

Correct options are (a) and (c)

Example 17: If the fourier expansion of function $f(x)$, $-a\le x\le a$, is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{a}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{a}\right),$$

then the value of b_2 for the function $f(x)$ which is shown in figure is _____.



Soln: Consider $f(x)$ in the interval $[-2, 2]$.

$$f(x)=\begin{cases}2+x & ; \quad -2 \le x < 0 \\ 2-x & ; \quad 0 \le x \le 2\end{cases}$$

Since, $f(-x)=f(x)$, $f(x)$ is an even function. Hence, $b_n = 0$. Therefore, $b_2 = 0$.

Correct answer is (0)

Example 18: By considering the fourier expansion of the function $f(x)=\pi+x,\ -\pi<x<\pi$, value of $\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)$ is equal to _____ $\times\pi$.

Soln: $f(x)=\pi+x,\ -\pi<x<\pi$

$$a_0=\frac{2}{2\pi}\int_{-\pi}^{\pi}(\pi+x)\,dx=\frac{1}{\pi}\left(\pi x+\frac{x^2}{2}\right)_{-\pi}^{\pi}=\frac{1}{\pi}\left(\pi^2+\frac{\pi^2}{2}+\pi^2-\frac{\pi^2}{2}\right)=2\pi$$

$$a_n=\frac{2}{2\pi}\int_{-\pi}^{\pi}(\pi+x)\cos(nx)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{\pi}\cos nx\,dx+\int_{-\pi}^{\pi}x\cos nx\,dx\right]=\frac{1}{\pi}\left[\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{\pi}+0\right]=0$$

$$b_n=\frac{2}{2\pi}\int_{-\pi}^{\pi}(\pi+x)\sin(nx)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{\pi}\pi\sin(nx)\,dx+\int_{-\pi}^{\pi}x\sin nx\,dx\right]$$

$$=\frac{1}{\pi}\left[0+x\times-\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}-\frac{\pi\cos n\pi}{n}\right]=-2\frac{(-1)^n}{n}$$

Hence, fourier series of $f(x)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx=\pi+\sum_{n=1}^{\infty}-\frac{2(-1)^n}{n}\sin nx$$

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$$\Rightarrow f(x)=\pi-2\left(-\sin x+\frac{\sin 2x}{2}-\frac{\sin 3x}{3}+\frac{\sin 4x}{4}-\frac{\sin 5x}{5}+\cdots\right)$$

$$\text{At } x=\frac{\pi}{2},\ \pi+\frac{\pi}{2}=\pi-2\left(-1+0+\frac{1}{3}+0-\frac{1}{5}+0+\frac{1}{2}\cdots\right)\ \Rightarrow\frac{\pi}{4}=\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)$$

Correct answer is (0.25)

Example 19: $f(x)$ is symmetric periodic function of x i.e. $f(-x)=f(x)$. Then, in general, the Fourier series of the function $f(x)$ will be of the form

(a) $f(x)=\sum_{n=1}^{\infty}\left(a_n\cos(nkx)+b_n\sin(nkx)\right)$

(b) $f(x)=a_0+\left(a_n\cos(nkx)\right)$

(c) $f(x)=\sum_{n=1}^{\infty}\left(b_n\sin(nkx)\right)$

(d) $f(x)=a_0+\sum_{n=1}^{\infty}\left(b_n\sin(nkx)\right)$

Soln: For a symmetric periodic function i.e., $f(-x)=f(x)$.
The Fourier series contains constant term and cosine terms.

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos(nkx)$$

Correct option is (b)

Example 20: A periodic function $f(x)$ of period 2π is defined in the interval $(-\pi<x<\pi)$ as:

$$f(x)=\begin{cases}-1 & \text{for} \quad -\pi<x<0\\ 1 & \text{for} \quad 0<x<\pi\end{cases}$$

The approximate Fourier series expansion for $f(x)$ is

(a) $f(x)=\frac{4}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+.....\right]$

(b) $f(x)=\frac{4}{\pi}\left[\sin x-\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x-.....\right]$

(c) $f(x)=\frac{4}{\pi}\left[\cos x+\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x+.....\right]$

(d) $f(x)=\frac{4}{\pi}\left[\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x-.....\right]$

Soln: $f(x)=\begin{cases}-1 & -\pi<x<0\\ 1 & 0>x>\pi\end{cases}$

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$$f(x)=\frac{a_0}{2}+\sum_n a_n\cos nx+\sum_n b_n\sin nx$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}-dx+\int_{0}^{\pi}dx\right]=\frac{1}{\pi}[-\pi+\pi]=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\cos nx dx+\int_{0}^{\pi}\cos nx dx\right]=\frac{1}{\pi}\left\{\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}+\left[\frac{\sin n\pi}{n}\right]_{0}^{\pi}\right\}=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\sin nx dx+\int_{0}^{\pi}\sin nx dx\right]$$

$$=\frac{1}{\pi}\left\{\left[\frac{\cos nx}{n}\right]_{-\pi}^{0}+\left[-\frac{\cos n\pi}{n}\right]_{0}^{\pi}\right\}=\frac{1}{\pi}\times\frac{1}{n}\left[\left(1-(-1)^n\right)+\left(1-(-1)^n\right)\right]$$

$$=\frac{2}{\pi}\left[\frac{1}{n}\left(1-(-1)^n\right)\right]$$

$$\Rightarrow \quad b_n = \begin{cases} 0 & \text{for even n} \\ \dfrac{4}{\pi n} & \text{for odd n} \end{cases}$$

$$\therefore\ f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots\ldots} \frac{1}{n} \sin nx = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\ldots \right)$$

Correct option is (a)

4.3 Half Range Series

Let the function is defined in the interval $(0, L)$ and it is insignificant whatever be the form of function outside $(0, L)$.

(1) To get cosine half range cosine series, we take $f(x)$ is an even function in the interval $(-L, L)$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0$$

(2) To get cosine half range sine series, we take $f(x)$ is an even function in the interval $(-L, L)$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = 0$$

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$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

4.4 Perseval's Formula

Full range Fourier series: $\displaystyle\int_{-l}^{l} [f(x)]^2\,dx = l\left\{\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right\}$

Half range cosine series : $\displaystyle\int_{0}^{l} [f(x)]^2\,dx = \frac{l}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$

Half range sine series: $\int_0^l [f(x)]^2 dx = \frac{l}{2}\left[\sum_{n=1}^{\infty} b_n^2\right]$

Example 21: By using the half range sine series for $f(x) = 1$ in $0 < x < \pi$, show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots.$$

Soln: $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi}(1)\sin nx\,dx = \frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)_0^{\pi} = \frac{-2}{n\pi}[\cos n\pi - 1] = \frac{-2}{n\pi}\left[(-1)^n - 1\right]$$

$= \frac{4}{n\pi}$ if n is odd

$= 0$ if n is even

Then the sine series is

$$1 = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \frac{4}{7\pi}\sin 7x + \ldots.$$

Using Perseval's formula, we get

$$\int_0^l \left[f(x)\right]^2 dx = \frac{l}{2}\left[b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + \ldots.\right]$$

$$\Rightarrow \int_0^{\pi}(1)^2 dx = \frac{\pi}{2}\left[\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \left(\frac{4}{5\pi}\right)^2 + \left(\frac{4}{7\pi}\right)^2 + \ldots.\right]$$

$$\Rightarrow [x]_0^{\pi} = \left(\frac{\pi}{2}\right)\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots.\right] \Rightarrow \pi = \frac{\pi}{2}\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots.\right]$$

$$\Rightarrow \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots.$$

Example 22: If $f(x) = \begin{cases} \pi x & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}$

Using half range cosine series, show that

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots..$$

Soln: $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx\right]$

$$= \pi\left(\frac{x^2}{2}\right)_0^1 + \pi\left(2x - \frac{x^2}{2}\right)_1^2 = \frac{\pi}{2} + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right] = \pi$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{2}\left[\int_0^1 \pi x\cos\frac{n\pi x}{2}dx + \int_1^2 \pi(2-x)\cos\frac{n\pi x}{2}dx\right]$$

$$= \pi\left[\frac{x\dfrac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - \left(\dfrac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)}\right]_0^1 + \pi\left[(2-x)\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - (-1)\left[\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right]\right]_1^2$$

$$= \pi\left[\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n2}\pi^2\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\right] + \pi\left[0 - \frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \pi\left[\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi\right] = \frac{4}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

$a_1 = 0, a_2 = -\frac{4}{\pi}, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = -\frac{4}{9\pi}\ldots.$

According to Perseval's formula:

$$\int_0^l [f(x)]^2 dx = \frac{l}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\right]$$

$$\Rightarrow \int_0^1 (\pi x)^2 dx + \int_1^2 \pi^2(2-x)^2 dx = \frac{2}{2}\left[\frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \ldots..\right]$$

$$\Rightarrow \pi^2\left[\frac{x^3}{3}\right]_0^1 - \pi^2\left[\frac{(2-x)^3}{3}\right]_1^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \ldots.. \Rightarrow \frac{\pi^2}{3} - \pi^2\left(0 - \frac{1}{3}\right) = \frac{\pi^2}{2} + \frac{16}{\pi^2}\left[1 + \frac{1}{81} + \ldots..\right]$$

$$\Rightarrow \frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots..\right] \Rightarrow \frac{\pi^2}{6} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots..\right]$$

$$\Rightarrow \frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots..$$

4.5 Fourier series in complex form

Suppose a function $f(x)$ is defined in the interval $(0, 2L)$

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{i\frac{n\pi x}{L}} + \sum_{n=1}^{\infty} c_{-n} e^{-i\frac{n\pi x}{L}}$$

$$c_0 = \frac{1}{2L}\int_0^{2L} f(x)dx; \quad c_n = \frac{1}{2L}\int_0^{2L} f(x)e^{-i\frac{n\pi x}{L}}dx; \quad c_{-n} = \frac{1}{2L}\int_0^{2L} f(x)e^{i\frac{n\pi x}{L}}dx$$

Example 23: Obtain the complex form of the Fourier series of the function

$$f(x)=\begin{cases}0, & -\pi\le x\le 0\\ 1, & 0\le x\le\pi\end{cases}$$

Soln: $c_0=\frac{1}{2\pi}\int_0^{\pi}dx=\frac{1}{2}$

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0}0.e^{-inx}dx+\int_0^{\pi}1.e^{-inx}dx\right]=\frac{1}{2\pi}\int_0^{\pi}e^{-inx}dx=\frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_0^{\pi}$$

$$=-\frac{1}{2n\pi i}\left[e^{-in\pi}-1\right]=-\frac{1}{2n\pi i}\left[\cos n\pi-i\sin n\pi-1\right]=-\frac{1}{2n\pi i}\left[(-1)^n-0-1\right]$$

$$c_n=\begin{cases}\frac{1}{in\pi}, & n \text{ is odd}\\ 0, & \text{n is even}\end{cases}$$

Similarly, $c_{-n}=\begin{cases}-\frac{1}{in\pi}, & n \text{ is odd}\\ 0, & \text{n is even}\end{cases}$

$$f(x)=\frac{1}{2}+\frac{1}{i\pi}\left[\frac{e^{ix}}{1}+\frac{e^{3ix}}{3}+\frac{e^{5ix}}{5}+.....\right]+\frac{1}{i\pi}\left[\frac{e^{-ix}}{-1}+\frac{e^{-3ix}}{-3}+\frac{e^{-5ix}}{-5}+.....\right]$$

$$=\frac{1}{2}-\frac{1}{i\pi}\left[\left(e^{ix}-e^{-ix}\right)+\frac{1}{3}\left(e^{3ix}-e^{-3ix}\right)+\frac{1}{5}\left(e^{5ix}-e^{-5ix}\right)+.....\right]$$

$$=\frac{1}{2}-\frac{2}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+.....\right]$$

Example 24: The complex form of fourier series of $f(x)$, where $f(x)=\left\{e^{-|x|},-2\le x\le 2\right.$ is

(a) $\sum_{n=-\infty}^{\infty}\frac{2}{4+n^2\pi^2}\left[1-(-1)^n e^{-2}\right]e^{i\frac{n\pi x}{2}}$ (b) $\sum_{n=-\infty}^{\infty}\frac{2}{4+n^2\pi^2}\left[1+(-1)^n e^{-2}\right]e^{in\pi x}$

(c) $\sum_{n=-\infty}^{\infty}\frac{2}{4+n^2\pi^2}\left[1-(-1)^n\right]e^{i\frac{n\pi x}{2}}$ (d) $\sum_{n=-\infty}^{\infty}\frac{2}{4+n^2\pi^2}\left[1+(-1)^n\right]e^{in\pi x}$

Soln: Complex form of fourier series is $f(x)=\sum_{-\infty}^{\infty}C_n e^{i\frac{2n\pi x}{4}}$

Here, $C_n=\frac{1}{4}\int_{-2}^{2}f(x)e^{-i\frac{2n\pi x}{4}}dx=\frac{1}{4}\int_{-2}^{0}e^x e^{-i\frac{n\pi x}{2}}dx+\frac{1}{4}\int_0^2 e^{-x}e^{-i\frac{n\pi x}{2}}dx$

$$=\frac{1}{4}\int_{-2}^{0}e^{\left(1-\frac{n\pi}{2}i\right)x}dx+\frac{1}{4}\int_0^2 e^{-\left(1+\frac{n\pi}{2}i\right)x}dx=\frac{1}{4}\left(\frac{e^{\left(1-\frac{n\pi i}{2}\right)x}}{1-\frac{n\pi}{2}i}\right)_{-2}^{0}+\frac{1}{4}\left(\frac{e^{-\left(1+\frac{n\pi i}{2}\right)x}}{-1+\frac{n\pi}{2}i}\right)_0^2$$

$$=\frac{1}{4}\left\{\frac{1}{1-\frac{n\pi}{2}i}\left(1-e^{-2\left(1-\frac{n\pi}{2}i\right)}\right)-\frac{1}{1+\frac{n\pi}{2}i}\left(e^{-2\left(1+\frac{n\pi}{2}i\right)}-1\right)\right\}$$

$$=\frac{1}{4}\left(\frac{1}{1-\frac{n\pi}{2}i}+\frac{1}{1+\frac{n\pi}{2}i}\right)+\frac{1}{4}\left(\frac{-1}{1-\frac{n\pi}{2}i}e^{-2}e^{n\pi i}-\frac{1}{1+\frac{n\pi}{2}i}e^{-2}e^{-n\pi i}\right)$$

$$=\frac{1}{4}\frac{2}{1+\frac{n^2\pi^2}{4}}+\frac{1}{4}\left(-\frac{1}{1-\frac{n\pi}{2}i}\cos n\pi-\frac{1}{1+\frac{n\pi}{2}i}\cos n\pi\right)e^{-2}$$

$$=\frac{2}{4+n^2\pi^2}-\left(\frac{2}{1+\frac{n^2\pi^2}{4}}\right)e^{-2}\cos n\pi\times\frac{1}{4}=\frac{2}{4+n^2\pi^2}\left[1-\cos n\pi\, e^{-2}\right]$$

$$=\frac{2}{4+n^2\pi^2}\left[1-(-1)^n e^{-2}\right]$$

$$\therefore\quad f(x)=\sum_{n=-\infty}^{\infty}\frac{2}{4+n^2\pi^2}\left[1-(-1)^n e^{-2}\right]e^{i\frac{n\pi x}{2}}.$$

Correct option is (a)

TIFR PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. A function $f(x)$ is defined in the range $-1\le x\le 1$ by

$$f(x)=\begin{cases}1-x & for\ \ x\ge 0\\ 1+x & for\ \ x<0\end{cases}$$

The first few terms in the Fourier series approximating this function are **[TIFR 2010]**

(a) $\frac{1}{2}+\frac{4}{\pi^2}\cos\pi x+\frac{4}{9\pi^2}\cos 3\pi x+\ldots$ (b) $\frac{1}{2}+\frac{4}{\pi^2}\sin\pi x+\frac{4}{9\pi^2}\sin 3\pi x+\ldots$

(c) $\frac{4}{\pi^2}\cos\pi x+\frac{4}{9\pi^2}\cos 3\pi x+\ldots$ (d) $\frac{1}{2}-\frac{4}{\pi^2}\cos\pi x+\frac{4}{9\pi^2}\cos 3\pi x-\ldots$

Soln. $f(x)=\begin{cases}1-x & x\ge 0\\ 1+x & x<0\end{cases}$ and $f(-x)=\begin{cases}1+x & x<0\\ 1-x & x\ge 0\end{cases}$

Hence, $f(x)$ is even $\quad(\because\ f(-x)=f(x))$

Therefore, $b_n=0$

$$a_0=\frac{2}{2}\int_{-1}^{1}f(x)\,dx=2\int_0^1(1-x)\,dx=2\left(x-\frac{x^2}{2}\right)_0^1=1$$

$$a_n=\frac{2}{2}\int_{-1}^{1}f(x)\cos\left(\frac{2\pi nx}{2}\right)dx=2\int_0^1(1-x)\cos(n\pi x)\,dx$$

$$=2\left[\int_0^1 \cos n\pi x\,dx-\int_0^1 x\cos n\pi x\,dx\right] \quad =2\left[\frac{\sin n\pi x}{n\pi}-\frac{x\sin n\pi x}{n\pi}-\frac{\cos n\pi x}{(n\pi)^2}\right]_0^1$$

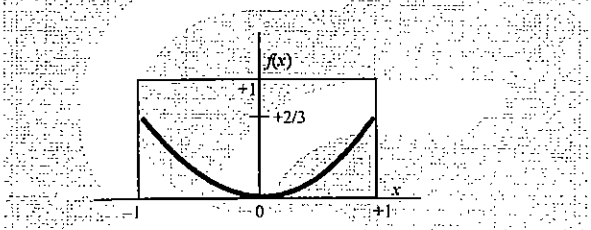
Therefore, $a_n=2\left[-\frac{\cos n\pi}{n^2\pi^2}+\frac{1}{n^2\pi^2}\right]=\begin{cases}\frac{2}{n^2\pi^2}[2] & n=\text{odd}\\ 0 & n=\text{even}\end{cases}$

Therefore, fourier series of $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi x$

$$\Rightarrow f(x)=\frac{1}{2}+\frac{4}{\pi^2}\left(\cos\pi x+\frac{\cos 3\pi x}{3^2}+\frac{\cos 5\pi x}{5^2}+\cdots\right).$$

Correct option is (a)

Question 2. A student is asked to find a series approximation for the function $f(x)$ in the domain $-1\le x\le +1$, as indicated by the thick line in the figure below.



The student represents the function by a sum of three terms

$$f(x)\approx a_0+a_1\cos\frac{\pi x}{2}+a_2\sin\frac{\pi x}{2}$$ **[TIFR 2013]**

Which of the following would be the best choices for the coefficients a_0, a_1 and a_2?

(a) $a_0=1, a_1=-\frac{1}{3}, a_2=0$ (b) $a_0=\frac{2}{3}, a_1=-\frac{2}{3}, a_2=0$

(c) $a_0=\frac{2}{3}, a_1=0, a_2=-\frac{2}{3}$ (d) $a_0=-\frac{1}{3}; a_1=0, a_2=-1$

Soln: Since, $f(x)$ is even (symmetric nature about x = 0), therefore $a_2=0$

At $x=0$; $f(x)=0\Rightarrow 0=a_0+a_1\Rightarrow a_0=-a_1$

Correct option is (b)

Question 1. The function $f(x)=\cosh x$ which exists in the range $-\pi \le x \le \pi$, is periodically repeated between $x=(2m-1)\pi$ and $(2m+1)\pi$ where $m=-\infty$ to ∞. Using Fourier series, undicate the correct relation a $x = 0$. **[JEST 2017]**

(a) $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{1-n^2}=\frac{1}{2}\left(\frac{\pi}{\cosh\pi}-1\right)$

(b) $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{1-n^2}=2\frac{\pi}{\cosh\pi}$

(c) $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{1+n^2}=2\frac{\pi}{\sinh\pi}$

(d) $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}=\frac{1}{2}\left(\frac{\pi}{\sinh\pi}-1\right)$

Soln: Since, $f(x)$ is even function, Hence $b_n = 0$

$$a_0=\frac{2}{2\pi}\int_{-\pi}^{\pi}\cosh x\,dx=\frac{2}{\pi}\int_0^{\pi}\cosh x\,dx=\frac{2}{\pi}(\sinh x)_0^{\pi}=\frac{2}{\pi}\sinh\pi$$

$$a_n=\frac{2}{2\pi}\int_{-\pi}^{\pi}f(x)\cos\left(\frac{2\pi nx}{2\pi}\right)dx=\frac{2}{\pi}\int_0^{\pi}\cosh x\cos(nx)\,dx$$

$$\text{Let } I=\int_0^{\pi}\cosh x\cos(nx)\,dx=\left(\cosh x\times\frac{\sin nx}{n}\right)_0^{\pi}-\int_0^{\pi}\sinh x\times\frac{\sin nx}{n}dx$$

$$=0-\left\{\left(\frac{1}{n}\sinh x\times-\frac{\cos nx}{n}\right)_0^{\pi}-\int_0^{\pi}\frac{1}{n}\cosh x\times-\frac{\cos nx}{n}dx\right\}$$

$$I=\frac{1}{n^2}\sinh\pi\cos n\pi-\frac{1}{n^2}I\Rightarrow\left(1+\frac{1}{n^2}\right)I=\frac{\sinh\pi\cos n\pi}{n^2}$$

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$$\text{Therefore, } a_n=\frac{2}{\pi}\left(\frac{\sinh\pi\cos n\pi}{n^2+1}\right)=\frac{2}{\pi}(-1)^n\left(\frac{\sinh\pi}{n^2+1}\right)$$

$$\text{Hence, } f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{2\pi nx}{2\pi}\right)\Rightarrow\cosh x=\frac{1}{2}\times\frac{2}{\pi}\sinh x+\sum_{-\infty}^{\infty}\frac{2}{\pi}\frac{\sinh\pi(-1)^n}{n^2+1}\cos nx$$

Putting $x = 0$ on both sides, we get

$$\Rightarrow 1=\frac{1}{\pi}\sinh\pi+\frac{2}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n\sinh\pi}{n^2+1}$$

$$\Rightarrow\frac{\pi}{2}=\sinh\pi\left(\frac{1}{2}+\sum_{-\infty}^{\infty}\frac{(-1)^n}{n^2+1}\right)\Rightarrow\sum_{-\infty}^{\infty}\frac{(-1)^n}{n^2+1}=\frac{1}{2}\left(\frac{\pi}{\sinh\pi}-1\right).$$

Correct option is (d)

DU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The Fourier series of a periodic function (with period 2π) is given by **[DU 2017]**

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

If $f(x)$ is defined by

$$f(x)=\begin{cases} x & \text{if } -\dfrac{\pi}{2}<x<\dfrac{\pi}{2} \\ \pi-x & \text{if } \dfrac{\pi}{2}<x<\dfrac{3\pi}{2} \end{cases}$$

then a_0 is equal to

(a) 0 (b) 2π (c) 1 (d) -2π

Soln: $$a_0=\frac{2}{\frac{3\pi}{2}-\left(-\frac{\pi}{2}\right)}\int_{-\pi/2}^{3\pi/2}f(x)\,dx=\frac{2}{2\pi}\int_{-\pi/2}^{\pi/2}x\,dx+\frac{2}{2\pi}\int_{\pi/2}^{3\pi/2}(\pi-x)\,dx=0+\frac{1}{\pi}\left(\pi x-\frac{x^2}{2}\right)_{\pi/2}^{3\pi/2}$$

$$=\frac{1}{\pi}\left[\pi\times\frac{3\pi}{2}-\frac{9\pi^2}{4\times2}-\pi\times\frac{\pi}{2}+\frac{\pi^2}{4\times2}\right]=\frac{\pi^2}{\pi}\left[\frac{3}{2}-\frac{9}{8}-\frac{1}{2}+\frac{1}{8}\right]=0$$

Correct option is (a)

HCU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The Fourier series of the functions **[HCU 2017]**

$$F(t)=\begin{cases}-1, & -\pi/\omega<t<0\\ +1, & 0<t<\pi/\omega\end{cases}$$

in the interval $-\pi/\omega<t<\pi/\omega$ can be written as

(a) $\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{\sin(n\omega t)}{n}$ (b) $\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{\cos(n\omega t)}{n}$

(c) $\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{\sin(n\omega t)}{n}+\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{\cos(n\omega t)}{n}$ (d) $\dfrac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\dfrac{\sin(n\omega t)}{n}$

Soln: $$f(t)=\frac{a_0}{2}+\sum_{n=0}^{\infty}a_n\cos\frac{2n\pi}{b-a}+\sum_{n=0}^{\infty}b_n\sin\left(\frac{2n\pi}{b-a}\right)$$

Since, $f(t)$ is an odd function

Therefore, $a_0=a_n=0$

$$b_n=\frac{2}{2\pi/\omega}\int_{-\pi/\omega}^{\pi/\omega}f(t)\sin\left(\frac{2\pi n}{2\pi/\omega}\right)dt \quad =\frac{\omega}{\pi}\left[\int_{-\pi/\omega}^{0}-\sin(n\omega t)\,dt+\int_{0}^{\pi/\omega}\sin(n\omega t)\,dt\right]$$

$$= \frac{\omega}{\pi} \times 2 \int_0^{\pi/\omega} \sin(n\omega t)\, dt = \frac{2\omega}{\pi n\omega} \left(\cos(n\omega t)\right)_0^{\pi/\omega} \quad = \frac{2}{\pi}\left[\cos(n\pi) - 1\right]$$

Therefore, $b_n = \begin{cases} \frac{2}{n\pi}[-1-1] & \text{if} \quad n = \text{odd} \\ \frac{2}{n\pi} \times 0 & \text{if} \quad n = \text{even} \end{cases}$

$\therefore \quad f(t) = \sum_{n=1,3,5,\ldots}^{\infty} -\frac{4}{n\pi} \sin(n\omega t).$

Correct option is (d)

BHU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The first term in the Fourier series of $f(x) = \frac{(\pi - x)^2}{4}$, in the range $(-\pi < x < \pi)$, is **[B.H.U-2016]**

(a) $\frac{\pi}{12}$ (b) $\frac{\pi^2}{6}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi^2}{12}$

Soln: $a_0 = \frac{2}{2\pi} \int_{-\pi}^{\pi} \frac{(\pi - x)^2}{4} dx = \frac{1}{4\pi} \int_{-\pi}^{\pi} (\pi^2 + x^2 - 2\pi x)\, dx = \frac{1}{4\pi}\left[\pi^2 x + \frac{x^3}{3} - 2\pi \frac{x^2}{2}\right]_{-\pi}^{\pi}$

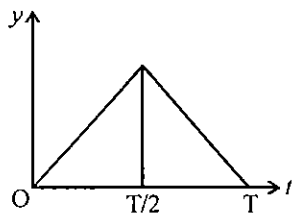
$= \frac{1}{4\pi}\left[\left(\pi^3 + \frac{\pi^3}{3} - \pi^3\right) - \left(-\pi^3 - \frac{\pi^3}{3} - \pi^3\right)\right] = \frac{1}{4\pi}\left[\frac{8\pi^3}{3}\right] = \frac{2\pi^2}{3}.$

Hence, 1st term $\frac{a_0}{2} = \frac{\pi^2}{3}$

Therefore, none of the options are correct.

FAST-FOR ENDEAVOUR

Question 2. Consider the following triangular periodic wave of period T.



The first term in the Fourier series of this triangular periodic wave is **[B.H.U-2016]**

(a) a (b) $\frac{a}{2}$ (c) $-a$ (d) $-\frac{a}{2}$

Soln: $a_0 = \frac{2}{T} \int_0^T f(t)\, dt = \frac{2}{T}[\text{area under triangle}] \quad = \frac{2}{T} \times \frac{1}{2} \times T \times a = a$

Hence, 1st term $\frac{a_0}{2} = \frac{a}{2}$.

Correct option is (b)

Chapter 5

COMPLEX ANALYSIS

5.1 Basic Review of Complex Numbers

Various representation of Complex number:

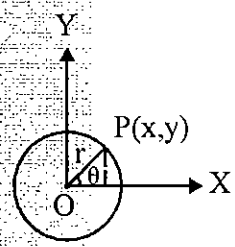
A complex number (z) is represented as $z = x + iy$, where x is the real part and y is imaginary part.

The conjugate of complex number z is represented by $\bar{z} = x - iy$

The modulus of complex number $z = |z| = r = \sqrt{x^2 + y^2}$ and argument of $z = Arg z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

The complex number (z) in polar form (r, θ) can also be written as

$$z = x + iy = r\, e^{i\theta} = r(\cos\theta + i\sin\theta)$$



where $x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow \quad x^2 + y^2 = r^2 = |z|^2$

This implies that $|z| = r$ represents a circle centred at the origin and having radius r.

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Properties of modulus of z:

If z_1 and z_2 are two complex numbers, then

(i) $|z_1 + z_2| \le |z_1| + |z_2|$ (ii) $|z_1 - z_2| \ge |z_1| - |z_2|$

(iii) $|z_1 z_2| = |z_1||z_2|$ (iv) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Properties of argument of z:

(i) $Arg(z_1.z_2.z_3.....z_n) = Arg.(z_1) + Arg.(z_2) + Arg.(z_3) + ... + Arg(z_n)$

(ii) $Arg\left(\frac{z_1}{z_2}\right) = Arg(z_1) - Arg(z_2)$

Some important Relations:

(i) $e^{i\theta} = \cos\theta + i\sin\theta$, (ii) $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, (iii) $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$

(iv) $\cos h\theta = \dfrac{(e^{\theta}+e^{-\theta})}{2}$, (v) $\sin h\theta = \dfrac{(e^{\theta}-e^{-\theta})}{2}$

De-Moiver's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Complex cube roots of unity:

$x^3 = 1 \Rightarrow x = 1, \omega = \dfrac{-1+i\sqrt{3}}{2}, \omega^2 = \dfrac{-1-i\sqrt{3}}{2}$ such that $1+\omega+\omega^2 = 0$ and $\omega^3 = 1$

Example 1: Multiplying a complex number z by $1+i$ rotates the radius vector of z by an angle of

(a) 90° clockwise (b) 45° anticlockwise
(c) 45° clockwise (d) 90° anticlockwise

Soln: $z = r\,e^{i\theta} \Rightarrow (1+i)z = \left(\sqrt{2}\,e^{i\pi/4}\right) r\,e^{i\theta} = \sqrt{2}\,r\,e^{i(\theta+\pi/4)}$

Multiplying (1 + i) with z rotates the radius vector anticlockwise by 45° and increases the modulus by a factor of $\sqrt{2}$.

Correct option is (a)

Example 2: If $z = (\lambda+3)+i\sqrt{5-\lambda^2}$ (λ is a real parameter and $i = \sqrt{-1}$), then the locus of z will be

(a) circle (b) ellipse (c) parabola (d) hyperbola

Soln: $z = (\lambda+3)+i\sqrt{5-\lambda^2} \Rightarrow$ real part $x = \lambda+3$ and imaginary part $y = \sqrt{5-\lambda^2}$

$\Rightarrow y^2 = 5-\lambda^2 = 5-(x-3)^2 \Rightarrow (x-3)^2 + y^2 = 5$ (Equation of circle)

Correct option is (a)

Example 3: If $z = x+iy$, then for e^z to be real.

(a) $y = n\pi,\ n = 0, \pm1, \pm2, \ldots.$ (b) $x = n\pi,\ n = 0, \pm1, \pm2, \ldots.$

(c) $y = (2n+1)\dfrac{\pi}{2},\ n = 0, \pm1, \pm2, \ldots.$ (d) $x = (2n+1)\dfrac{\pi}{2},\ n = 0, \pm1, \pm2, \ldots.$

Soln: $e^z = e^{x+iy} = e^x\,e^{iy} = e^x(\cos y + i\sin y)$

Hence, for e^z to be real, $\sin y = 0 \Rightarrow y = n\pi,\ n = 0, \pm1, \pm2, \ldots.$

Correct option is (a)

Example 4: If $z = x+iy$, then $\mathrm{Im}\left(|z-4i|^2 + 3\bar{z}\right)$ is equal to

(a) $3y$ (b) $-3y$ (c) $(y-4)$ (d) $-(y-4)$

Soln: $|z-4i|^2+3\bar{z}=|x+iy-4i|^2+3(x-iy) \quad =x^2+(y-4)^2+3x-3iy$

$=(x^2+(y-4)^2+3x)-i3y$

Hence, imaginary part of $\left(|z-4i|^2+3\bar{z}\right)$ is $-3y$

Correct option is (b)

Example 5: $\left(1+\sqrt{3}\,i\right)^{1+i}$ can be expressed as

(a) $\left(1+\sqrt{3}\,i\right)e^{-\pi/3}\left[\cos(\log 2)+i\sin(\log 2)\right]$ (b) $\left(1+\sqrt{3}\,i\right)e^{\pi/3}\left[\cos(\log 2)-i\sin(\log 2)\right]$

(c) $(1+i)e^{\pi/3}\left[\cos\left(\log\sqrt{3}\right)+i\sin\left(\log\sqrt{3}\right)\right]$ (d) None of these

Soln: $\left(1+\sqrt{3}\,i\right)^{1+i}=e^{(1+i)\log\left(1+\sqrt{3}i\right)}=e^{(1+i)(\log 2+i(\pi/3))}$

$=e^{\left(\log 2-\frac{\pi}{3}\right)+i\left(\log 2+\frac{\pi}{3}\right)} \quad =e^{\log 2}e^{-\pi/3}\times e^{i\log 2}e^{i\frac{\pi}{3}}$

$=2e^{-\pi/3}\left[\cos(\log 2)+i\sin(\log 2)\right]\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]$

$=\left(1+\sqrt{3}\,i\right)e^{-\pi/3}\left[\cos(\log 2)+i\sin(\log 2)\right].$

Correct option is (a)

Example 6: If z is a complex number such that $z=x+iy$, then z will be either purely real or purely imaginary if and only if z^2 is equal to

(a) $(\bar{z})^2$ (b) $(\bar{z})$ (c) $|\bar{z}|^2$ (d) None of these

Soln: If $z^2=(\bar{z})^2 \Rightarrow (x+iy)^2=(x-iy)^2 \quad \Rightarrow (x+iy)^2-(x-iy)^2=0$

$\Rightarrow (x+iy+x-iy)(x+iy-x+iy)=0 \quad \Rightarrow (2x)(2iy)=0$

Hence, either $x=0$ or $y=0$.

Hence, z is purely imaginary or purely real accordingly.

Correct option is (a)

Example 7: Principle value of $\arg(z)$, where $z=-\dfrac{\sqrt{2}\,i}{1+i}$ is

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $-\frac{3\pi}{4}$ (d) $-\frac{\pi}{4}$

Soln: $z=-\dfrac{\sqrt{2}\,i}{1+i}=-\dfrac{\sqrt{2}\,i}{1+i}\times\dfrac{1-i}{1-i} \quad =-\dfrac{\sqrt{2}\,(1+i)}{2}=-\dfrac{1}{\sqrt{2}}-\dfrac{i}{\sqrt{2}}$

$\arg(z)=\tan^{-1}\left(\dfrac{-1/\sqrt{2}}{-1/\sqrt{2}}\right)=\dfrac{\pi}{4}$

Since, z lies in 3rd quadrant. Hence, principle value of $\arg(z)$ is $= \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$.

Correct option is (c)

Example 8: If $|z| = 1$, then for all complex numbers 'a' and 'b' value of $\left|\frac{az+b}{\bar{b}z+\bar{a}}\right|$ is equal to

(a) $\sqrt{a^2+b^2}$ (b) 1 (c) $\frac{a}{b}$ (d) $\frac{b}{a}$

Soln: $\left|\frac{az+b}{\bar{b}z+\bar{a}}\right| = \frac{\left|z\left(a+\frac{b}{z}\right)\right|}{\left|\bar{b}z+\bar{a}\right|} = \frac{|z|\left|a+bz^{-1}\right|}{\left|\bar{b}z+\bar{a}\right|}$ $\left[\because |z|=1,\ z\bar{z}=1,\ \bar{z}=z^{-1}\right]$

$= \frac{|a+b\bar{z}|}{|\bar{b}z+\bar{a}|}$

Now, since $|\bar{z}| = |z|$

$|a+b\bar{z}| = \overline{|a+b\bar{z}|} = |\bar{a}+\bar{b}\bar{\bar{z}}| = |\bar{a}+\bar{b}z|$

$\therefore \left|\frac{az+b}{\bar{b}z+\bar{a}}\right| = \frac{|a+b\bar{z}|}{|\bar{b}z+\bar{a}|} = \frac{|\bar{a}+\bar{b}z|}{|\bar{a}+\bar{b}z|} = 1.$

Correct option is (b)

Example 9: $\frac{1+i}{\sqrt{3}-i}$ is equal to

(a) $\sqrt{2}\,e^{i\frac{5\pi}{12}}$ (b) $\frac{1}{\sqrt{2}}e^{i\frac{5\pi}{12}}$ (c) $\sqrt{2}\,e^{i\frac{5\pi}{4}}$ (d) $\frac{1}{\sqrt{2}}e^{i\frac{5\pi}{4}}$

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Soln: We have $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = \sqrt{2}\,e^{i\frac{\pi}{4}}$ and $\sqrt{3}-i = 2\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right) = 2e^{i\frac{\pi}{6}}$

Hence, $\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}\,e^{i\frac{\pi}{4}}}{2e^{-i\frac{\pi}{6}}} = \frac{1}{\sqrt{2}}e^{i\frac{5\pi}{12}}$.

Correct option is (b)

Example 10: For the roots of unity $z = e^{2\pi\frac{i}{m}}$, $m>1$. The value of $1+z+z^2+\cdots z^{m-1}$ is equal to _____.

Soln: $1+z+z^2+\cdots z^{m-1} = \frac{1-z^m}{1-z}$ (sum of 'm' terms of G.P.)

Here, $z^m = e^{2\pi\frac{i}{m}\times m} = e^{2\pi i} = 1$

Therefore, $\frac{1-z^m}{1-z}=0$.

Correct answer is (0)

Example 11: Imaginary part of $\sin^{-1}\left(\frac{3i}{4}\right)$ is _____.

Soln: Let $z=x+iy$

$$\sin^{-1}\left(\frac{3i}{4}\right)=z \Rightarrow \sin z=\frac{3i}{4} \Rightarrow \frac{e^{iz}-e^{-iz}}{2i}=\frac{3i}{4} \Rightarrow e^{iz}-e^{-iz}=-\frac{3}{2}$$

Let $e^{iz}=t$. Hence, $t-\frac{1}{t}=-\frac{3}{2} \Rightarrow t^2+\frac{3t}{2}-1=0$

Therefore, $t=\frac{-\frac{3}{4}\pm\sqrt{\frac{3}{4}-4\times-1}}{2} \Rightarrow t=-\frac{3}{4}\pm\frac{5}{4} \Rightarrow t=\frac{1}{2}$

$$\Rightarrow e^{iz}=\frac{1}{2} \Rightarrow iz=\log\left(\frac{1}{2}\right) \Rightarrow z=i\log 2$$

Hence, imaginary part $=\log 2=0.693$.

Correct answer is (0.693)

Example 12: If $|z|=1$, and $|z^2+2z+6+8i|\le n$, then value of n is _____. (Here, $z=x+iy$)

Soln: We have, $|z_1+z_2+z_3+\cdots z_n|\le|z_1|+|z_2|+|z_3|+\cdots|z_n|$

Hence, $|z^2+2z+6+8i|\le|z^2|+2|z|+|6+8i|\le|z|^2+2|z|+|6+8i|\le 1+2\times 1+\sqrt{36+64}\le 13$

Therefore, $n=13$.

Correct answer is (13)

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5.2 Function of A complex Variable

Basic Representation:

$$f(z)=u(x,y)+iv(x,y)$$

Example: $f(z)=z^2=(x+iy)^2=\underbrace{(x^2-y^2)^2}_{u(x,y)\text{ is real part}}+\underbrace{i2xy}_{v(x,y)\text{ is imaginary part}}$

Existance of $\lim_{z\to z_0} f(z)$:

The limit will exists only if the limiting value is independent of the path along which z approaches z_0

Example 13: Find whether the limit $\lim_{z\to 0}\frac{z}{|z|}$ exist or not.

Soln: $z \to 0$ means $x \to 0$ & $y \to 0$

For z = 0, we have to choose a path passing through a origin.

Therefore, we have choosen a straight line passing through the origin i.e. y = mx

$$\lim_{z\to 0}\frac{z}{|z|}=\lim_{\substack{z\to 0\\ y\to 0}}\frac{x+iy}{\sqrt{x^2+y^2}}=\lim_{x\to 0}\frac{x+imx}{\sqrt{x^2+m^2x^2}}=\frac{1+im}{\sqrt{1+m^2}}$$

Therefore, the limit depends on m i.e. slope of the straight line. Thus, the limiting values is dependent on the path and the limit does not exists.

Example 14: Calculate the value $\lim_{z\to\infty}\frac{iz^3+iz-1}{(2z+3i)(z-i)^2}$

Soln: $\lim_{z\to\infty}\frac{iz^3+iz-1}{(2z+3i)(z-i)^2}=\lim_{z\to\infty}\frac{z^3\left(i+\frac{i}{z^2}-\frac{1}{z^3}\right)}{z\left(2+\frac{3i}{z}\right)z^2\left(1-\frac{1}{z}\right)^2}=\frac{i}{2}$

Differentiability of complex function:

$$f'(z)=\underset{\delta z\to 0}{Lt}\frac{[f(z+\delta z)-f(z)]}{\delta z}$$

The function will be differentiable if limit exists i.e the limiting value will be independent of path along with $\delta z \to 0$.

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Example: $f(z) = (4x + y) + i(4y - x) \Rightarrow u = (4x + y)$ and $v = (4y - x)$

$\Rightarrow f(z + \delta z) = 4(x + \delta x) + (y + \delta y) + i\,[4(y + \delta y) - (x + \delta x)]$

$\Rightarrow f(z + \delta z) - f(z) = 4\delta x + \delta y + i(4\delta y - \delta x)$

$$\Rightarrow \frac{f(z+\delta z)-f(z)}{\delta z}=\frac{4\delta x+\delta y+i(4\delta y-\delta x)}{\delta z}\Rightarrow\frac{\delta f}{\delta z}=\frac{4\delta x+\delta y-i\delta x+4i\delta y}{\delta x+i\delta y}$$

Along real axis : $\delta x = \delta z$, $\delta y = 0$, $\Rightarrow \frac{\delta f}{\delta z} = 4 - i$

Along imaginary axis : $i\delta y = \delta z$, $\delta x = 0$, $\Rightarrow \frac{\delta f}{\delta z} = 4 - i$

Along a line : $y = x$, $\delta y = \delta x$, $\delta z = (1 + i)\,\delta x$, $\Rightarrow \frac{\delta f}{\delta z}=\frac{5\delta x+3i\delta x}{(1+i)\delta x}=\frac{5+3i}{1+i}=4-i$

Therefore, $f(z) = (4x + y) + i(4y - x)$ will be differentiable.

5.3 Complex Analytic Function

A function $f(z)$ is said to be analytic at a point $z = z_0$ if it is defined and differentiable at every point in some neighhourhood of z_0. The function $f(z)$ is said to be analytic in a domain D if it is defined and is differentiable at every point of domain D.

Cauchy Reamann Equations in Cartesian Coordinates:

For a function $f(z) = u + iv$ to be analytic at all points in some region R,

Necessary conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Sufficient Condition: $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y.

Derivative of $f(z)$: $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

Example 15: Check whether $f(z) = \sin z$ is analytic or not.

Soln: $f(z) = \sin z = \sin(x+iy) = \sin x.\cos(iy) + \cos x.\sin(iy) = \sin x.\cosh y + i\cos x.\sinh y$

Therefore, u = sinx.coshy and v = cosx.sinhy

$\frac{\partial u}{\partial x} = \cos x.\cosh y; \frac{\partial u}{\partial y} = \sin x.\sinh y; \frac{\partial v}{\partial x} = -\sin x.\sinh y; \frac{\partial v}{\partial y} = \cos x.\cosh y$

So, C-R equation is satisfied, given f(z) is analytic.

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Example 16: The harmonic conjugate function of $u(x,y) = 2x(1-y)$ corresponding to a complex analytic function $\omega = u(x,y) + iv(x,y)$ is given $v(x,y) = \alpha x^2 + \beta y + \gamma y^2$ (Taking the integration constant to be zero). Which of the following statement is true ?

(a) $\alpha - \gamma = \beta$ (b) $\alpha + \gamma + \beta = 0$ (c) $\alpha + \gamma = \beta$ (d) $\alpha\gamma\beta = 1$

Soln: $u(x,y) = 2x(1-y)$

$$\Rightarrow \frac{\partial u}{\partial x} = 2(1-y) = \frac{\partial v}{\partial y} \Rightarrow v = 2y - y^2 + f_1(x)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -2x = -\frac{\partial v}{\partial x} \Rightarrow v = x^2 + f_2(y)$$

Therefore, the imaginary part of the complex function $v = x^2 - y^2 + 2y$

Comparing with the question, $\alpha = 1, \beta = 2, \gamma = -1 \Rightarrow \alpha - \gamma = \beta$

Correct answer is (a)

Example 17: Let $u(x,y) = \dot{x} + \frac{1}{2}(x^2 - y^2)$ be the real part of an analytic function $f(z)$ of the complex variable $z = x + iy$. Then the imaginary part of $f(z)$ is

(a) $y + xy$ (b) xy (c) y (d) $y^2 - x^2$

Soln: $u(x,y) = x + \frac{1}{2}(x^2 - y^2)$; $f(z) = u(x,y) + iv(x,y)$

Cauchy Reamann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial x} = 1 + x \Rightarrow \frac{\partial v}{\partial y} = 1 + x$

$\Rightarrow \quad v = y + xy + g(x)$ ... (1)

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \frac{\partial u}{\partial y} = -y \Rightarrow \frac{\partial v}{\partial x} = y \Rightarrow v = xy + h(y)$ ... (2)

Compare (1) & (2), taking $h(y) = y$; and $g(x) = 0$; $v = y + xy$

Correct answer is (a).

Cauchy Reamann equations in Polar co-ordiantes:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Derivative of $f(z)$: $f'(z) = (\cos\theta - i\sin\theta)\frac{\partial f}{\partial r} = -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial f}{\partial \theta}$

5.4 Power Series Expansion of Complex Function

Every complex function which is analytic at $z = z_0$ can be expanded into power series about $z = z_0$. Power series expansion of $f(z)$ about $z = z_0$, is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$$

where, z_0 is the centre of power series.

Radius of convergence of power series:

Consider a circle centered at $z = z_0$ and radius R i.e. $|z - z_0| = R$, such that the power series is convergent for all points of the region $|z - z_0| < R$ (i.e. within the circle) and divergent for some specific points of the region $|z - z_0| > R$ (outside the circle). Therefore, R is said to be the radius of convergence of power series and defined as

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

and $|z - z_0| < R$ is said to be the region of convergence of the power series.

Example 18: Calculate the radius of convergence and region of convergence for the following power series

$$\sum_{n=0}^{\infty}\frac{(2n+3)}{(2n+5)(n+5)}z^n$$

Soln: For the given power series, $a_n=\frac{(2n+3)}{(2b+5)(n+5)}$, $z_0=0$

$$R=\lim_{n\to\infty}\left|\frac{2n+3}{(2n+5)(n+5)}\cdot\frac{(2(n+1)+5)(n+1+5)}{2(n+1)+3}\right|=\lim_{n\to\infty}\left|\frac{\left(2+\frac{3}{n}\right)\left(2+\frac{7}{n}\right)\left(1+\frac{6}{n}\right)}{\left(2+\frac{5}{n}\right)\left(1+\frac{5}{n}\right)\left(2+\frac{5}{n}\right)}\right|=1$$

Region of convergence $|z|<1$

This series is convergent within a circle of radius 1 and centre (0, 0).

Example 19: Find the radius of convergence of the series

$$\frac{z}{2}+\frac{1.3}{2.5}z^2+\frac{1.3.5}{2.5.8}z^3+.....$$

Soln: The coefficient of z^n of the given power series is given by

$$a_n=\frac{1.3.5...(2n-1)}{2.5.8...(3n-1)}$$

$$a_{n+1}=\frac{1.3.5...(2n-1)(2n+1)}{25.8...(3n-1)(3n+2)}$$

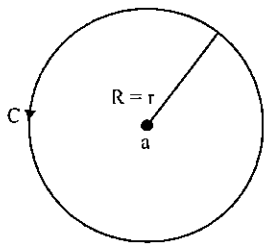
So, $$\frac{a_{n+1}}{a_n}=\frac{2n+1}{3n+2}=\frac{2}{3}\cdot\frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{2}{3n}\right)}\Rightarrow R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{3}{2}\cdot\frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{2}{3n}\right)}\right|=\frac{3}{2}$$

Taylor Series Expansion

If a function $f(z)$ is analytic at all points inside the circle C, having center at $z=a$ and radius r, then at each point z inside C, the Taylor series expansion of $f(z)$ about $z=a$, is

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+....+\frac{f^n(a)}{n!}(z-a)^n+....$$

$$\Rightarrow f(z)=\sum_{n=0}^{\infty}\frac{f^n(z_0)}{n!}(z-z_0)^n$$



Example 20: Expand the function $f(x) = \sin x$ about the point $x = \frac{\pi}{6}$.

Soln: $f(x) = f\left(\frac{\pi}{6}\right) + \left(x - \frac{\pi}{6}\right) f'\left(\frac{\pi}{6}\right) + \frac{1}{2!}\left(x - \frac{\pi}{6}\right)^2 f''\left(\frac{\pi}{6}\right) + \ldots\ldots$

$$= \sin\frac{\pi}{6} + \left(x - \frac{\pi}{6}\right)\cos\frac{\pi}{6} + \frac{\left(x - \frac{\pi}{6}\right)^2}{2!}\left(-\sin\frac{\pi}{6}\right) + \ldots\ldots$$

$$= \frac{1}{2} + \left(x - \frac{\pi}{6}\right)\frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right)^2\frac{1}{2} - \frac{1}{6}\left(x - \frac{\pi}{6}\right)^3\frac{\sqrt{3}}{2} + \ldots\ldots$$

$$f(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{1}{4\sqrt{3}}\left(x - \frac{\pi}{6}\right)^3 + \ldots.$$

Example 21: Expand the following function $f(z) = \frac{1}{(z-1)(z-4)}$ about z = 2.

Soln: $\frac{1}{(z-1)(z-4)} = \frac{1}{3}\frac{1}{z-4} - \frac{1}{3}\frac{1}{z-1} = \frac{1}{3}\frac{1}{(z-2)-2} - \frac{1}{3}\frac{1}{(z-2)+1}$

$$= \frac{1}{3(-2)}\frac{1}{\left[1 - \frac{z-2}{2}\right]} - \frac{1}{3}\frac{1}{\left[1 + (z-2)\right]} = -\frac{1}{6}\left[1 - \frac{z-2}{2}\right]^{-1} - \frac{1}{3}\left[1 + (z-2)\right]^{-1}$$

$$= -\frac{1}{6}\left[1 + \frac{z-2}{2} + \frac{(z-2)^2}{4} + \frac{(z-2)^3}{8} \ldots\ldots\right] - \frac{1}{3}\left[1 - (z-2) + (z-2)^2 - (z-2)^3 + \ldots..\right]$$

$$= -\frac{1}{2} + \frac{1}{4}(z-2) - \frac{3}{8}(z-2)^2 + \ldots\ldots$$

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Example 22: The first few terms in the Taylor series expansion of the function f(x) = sinx around $x = \frac{\pi}{4}$ are

(a) $\frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right) + \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \ldots\ldots\ldots\right]$

(b) $\frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right) + \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \ldots\ldots\ldots\right]$

(c) $\left[\left(x - \frac{\pi}{4}\right) - \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \ldots\ldots\ldots\right]$

(d) $\frac{1}{\sqrt{2}}\left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\right]$

Soln: $f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)$

$f(x)=\sin x,\quad f'(x)=\cos x,\quad f''(x)=-\sin x,\qquad f'''(x)=-\cos x$

$f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}},\quad f'\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}},\quad f''\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

$\therefore\qquad f(x)=\frac{1}{\sqrt{2}}\left[1+\left(x-\frac{\pi}{4}\right)-\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2-\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3+.....\right]$

Correct answer is (b)

Example 23: The taylor series expansion of the function $\ln(\cosh x)$, where x is real, about point $x=0$ starts with the following terms:

(a) $-\frac{1}{2}x^2+\frac{1}{12}x^4+.....$ (b) $\frac{1}{2}x^2-\frac{1}{12}x^4+.....$

(c) $-\frac{1}{2}x^2+\frac{1}{6}x^4+.....$ (d) $\frac{1}{2}x^2+\frac{1}{6}x^4+.....$

Soln: Taylor series expansion of $f(x)$ about $x=x_0$ is:

$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)+.....$

Now, in question $x_0=0$, $f(x)=\ell n(\cosh x)$

$f'(x)=\frac{1}{\cosh x}\sinh x=\tanh x$

$f''(x)=\sec h^2x;$

$f'''(x)=2\text{sech } x(-\text{sech } x.\tanh x)=-2\text{sech}^2 x.\tanh x$

$f''''(x)=-2(2\text{sech } x)(-\text{sech } x.\tanh x)\tanh x-2\text{sech}^2 x.\text{sech}^2 x=4\text{sech}^2 x.\tanh^2 x-2\text{sech}^4 x$

Required Taylor series expansion: $\ell n(\cosh x)=0+0+\frac{1}{2}.1.x^2+0+\frac{1}{4!}(-2)x^4+.....$

$\Rightarrow\qquad \ell n(\cosh x)=\frac{1}{2}x^2-\frac{1}{12}x^4+.....$

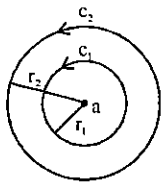
Correct answer is (b)

Laurent Series expansion:

Laurent's Theorem:

If $f(z)$ is analytic inside the ring-shaped region R bounded by two concentric circles C_1 and C_2 with centre at $z=a$ and respective radii r_1 and r_2 $(r_2>r_1)$, then for all points z in the region R, the Laurent series expansion of $f(z)$ about $z=a$ will be

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$



where, $a_n=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-a)^{n+1}}dz$

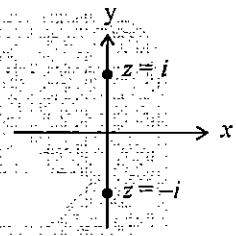
$$b_n=\frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{(z-a)^{-(n+1)}}dz$$

Note: (i) The Laurrent series converges in the region R.

(ii) The point at which Laurrent series expansion has to be calculated, will be the centre of concentric circles.

Example 24: $f(z)=\frac{1}{1+z^2}$ about $z=i$

Soln: $f(z)=\frac{1}{1+z^2}=\frac{1}{(z+i)(z-i)}=\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]$



$$=\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{(z-i)+2i}\right]$$

$$=\frac{1}{2i}(z-i)^{-1}-\frac{1}{2i}\left[\frac{1}{2i\left[1+\frac{(z-i)}{2i}\right]}\right]=\frac{1}{2i}(z-i)^{-1}-\frac{1}{4i^2}\left[1-\frac{(z-i)}{2i}+\frac{(z-i)^2}{4i^2}.....\right]$$

$$=\underset{\text{Converges for } |z-i|\geq 0}{\frac{1}{2i}(z-i)^{-1}}+\underset{\text{Converges for } |z-i|<2}{\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{(z-i)^n}{(2i)^n}}$$

So, the function will be convergent for $0\leq|z-i|<2$

Example 25: Exapand the function $f(z)=\frac{1}{(z-1)(z-2)}$ in the annulus $z=1$ and $z=2$.

Soln: $f(z)=\frac{1}{(z-1)(z-2)}=\frac{1}{(z-2)}-\frac{1}{(z-1)}$ [when $1<|z|<2$, then $\frac{1}{|z|}<1$ and $\frac{|z|}{2}<1$]

$$f(z)=\frac{1}{-2\left(1-\frac{z}{2}\right)}-\frac{1}{z\left(1-\frac{1}{z}\right)}=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$=-\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+....\right]-\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+.....\right]=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Example 26: Find the Laurent series of the function $f(z)=\frac{1}{z^2(1-z)}$ about z=0

Soln: $f(z)=\frac{1}{z^2(1-z)}=\frac{1}{z^2}(1-z)^{-1}=\frac{1}{z^2}(1+z+z^2+.....)=\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+.....$

Example 27: Expand $\frac{1}{z^2-3z+2}$ for (i) $0<|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$.

Soln: $\frac{1}{z^2-3z+2}=\frac{1}{(z-1)(z-2)}=\frac{1}{(z-2)}-\frac{1}{(z-1)}$

(i) $0<|z|<1$: then $\frac{|z|}{2}<1$ and $|z|<1$

$$f(z)=\frac{1}{-2\left(1-\frac{z}{2}\right)^{-1}}-\frac{1}{-1(1-z)}=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}+(1-z)^{-1}$$

$$=-\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+.....\right]+\left[1+z+z^2+.....\right]=\frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n+\sum_{n=0}^{\infty}z^n$$

(ii) $1<|z|<2$: then $\frac{1}{|z|}<1$ and $\frac{|z|}{2}<1$

$$\therefore\quad f(z)=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+.....\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+.....\right)$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

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(iii) $|z|>2$: then $\frac{2}{|z|}<1$ and $\frac{1}{|z|}<1$

$$\therefore\quad f(z)=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+.....\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+.....\right)$$

$$=\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Example 28: The Laurent series expansion of the function $f(z)=e^z+e^{1/z}$ about $z=0$ is given by

(a) $\sum_{n=-\infty}^{\infty}\frac{z^n}{n!}$ for all $|z|<\infty$

(b) $\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n!}$ only if $0<|x|<1$

(c) $\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n!}$ for all $0<|z|<\infty$

(d) $\sum_{n=-\infty}^{\infty}\frac{z^n}{n!}$, only if $|z|<1$

Soln: Given the complex function: $f(z)=e^{z}+e^{1/z}$

$$=\left[1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+.....\right]+\left[1+\frac{1}{z}+\frac{1}{z^2.2!}+\frac{1}{z^3.3!}+.....\right]$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{n!}+\sum_{n=0}^{\infty}\frac{1}{z^n}\frac{1}{n!}=\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n!}$$

ince, e^{z} is not analytic at $z=\infty$ and $e^{1/z}$ is not analytic at $z=0$,

Therefore, the laurrent series expansion will be convergent for $0<|z|<\infty$

Correct Option is (c)

5.5 Singularity of Complex Function

Singular points of a function:

The points at which the function ceases to be analytic, are said to be the singular points of the function.

Example: $f(z)=\frac{1}{(z-2)}$ has a singularity at z = 2.

The singularity will be of the following types:

Isolated singularity:

A point $z=z_0$ is said to be isolated singularity of $f(z)$ if

(a) $f(z)$ is not analytic at $z=z_0$

(b) $f(z)$ is analytic in the neighbourhood of $z=z_0$ i.e. there exists a neighbourhood of $z=z_0$, containing no other singularity.

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Example:

(i) Function $f(z)=\frac{1}{z}$ is analytic everywhere except at z = 0, therefore z = 0 is an isolated singularity.

(ii) The function $f(z)=\frac{z+2}{(z-1)(z-2)(z-3)}$ has three isolated singularities at $z=1, 2$ and 3.

Non-isolated singularity:

A point $z=z_0$ is said to be non-isolated singularity of $f(z)$ if

(a) $f(z)$ is not analytic at $z=z_0$

(b) there exists a neighbourhood of $z=z_0$, containing other singularities of $f(z)$.

Example:

$$f(z)=\frac{1}{\left[\sin\frac{\pi}{z}\right]}$$

Condition of singularity: $\sin\frac{\pi}{z}=0\Rightarrow\frac{\pi}{z}=n\pi\Rightarrow z=\frac{1}{n}\,(n=0,1,2,.....\infty)$

The point $z=0$ corresponding to $n\to\infty$, will be surrounded by infinite many singular points of $f(z)$. Thus, $z=0$ is an non-isolated singularity of $f(z)$.

Types of isolated singularity

If $f(z)$ is an isolated singular point at $z=a$, then we can expand $f(z)$ about $z=a$ into Laurent series as:

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\underbrace{\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}}_{\text{principal part of the expansion}}=\left[a_0+a_1(z-a)+a_2(z-a)^2+.....\right]+\left[\frac{b_1}{(z-a)}+\frac{b_2}{(z-a)^2}+.....\right]$$

Therefore, three types of singulatiry are as follows:

(1) Removable singularity:

If the principal part of the Laurrent series expansion of $f(z)$ about $z=a$ contains no term i.e. $b_n=0$ for all 'n', then $f(z)$ has a removable singularity at $z=a$.

In this case, Laurent series expansion is $f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n$

Example: $f(z)=\frac{\sin z}{z}=\frac{1}{z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-.....\right)=1-\frac{z^2}{3!}+\frac{z^4}{5!}-.....$

Since, there is no negative term in the laurent series expansion of $f(z)$ about $z=0$, hence $z=0$ is a removable singularity of $f(z)$.

(2) Non-essential singularity or Pole:

If the principal part of the Laurrent series expansion of $f(z)$ about $z=a$ contains finite number of terms, say m, i.e. $b_n=0$ for all $n>m$, then $f(z)$ has a non-essential singularity or a pole of order m at $z=a$. A pole of order one is also known as simple pole.

Thus if $z=a$ is pole of order m of function $f(z)$, then $f(z)$ will have the Laurent series expansion of the form

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{m}b_n(z-z_0)^{-n}$$

Example: $f(z)=\frac{z}{(z-1)(z+2)^2}$ has a simple pole at $z=1$ and a pole of order 2 at $z=-2$.

(3) Essential singularity:

If the principal part of the Laurrent series expansion of $f(z)$ about $z=a$, contains infinite number of terms i.e. $b_n \neq 0$ for infinitely many values of n, then $f(z)$ has an essential singularity at $z=a$.

Example: $f(z)=e^{1/z}=1+\frac{1}{z}+\frac{1}{z^2\,2!}+\ldots\ldots$

Therefore, $f(z)$ has an essential singularity at $z=0$.

Examples of Singular Points:

(1) $f(z)=\frac{1}{(z-1)(z+3)}=\frac{1}{4}\frac{1}{(z-1)}-\frac{1}{4}\frac{1}{(z+3)}$

For Laurent series expansion about $z=1$

$$f(z)=\frac{1}{4}\frac{1}{(z-1)}-\frac{1}{4}\frac{1}{(z-1)+4}=\frac{1}{4}\frac{1}{(z-1)}-\frac{1}{4}\frac{1}{4\left[1+\frac{(z-1)}{4}\right]}$$

$$=\frac{1}{4}\frac{1}{(z-1)}-\frac{1}{16}\left[1+\frac{(z-1)}{4}\right]^{-1}=\frac{1}{4}\cdot\frac{1}{(z-1)}-\frac{1}{16}\left[1-\frac{(z-1)}{4}+\frac{(z-1)^2}{16}-\ldots\ldots\right]$$

$$=\frac{1}{4(z-1)}-\frac{1}{16}+\frac{(z-1)}{64}-\frac{(z-1)^2}{256}+\ldots\ldots$$

Therefore, $z=1$ is a pole of order $=1$

Similarly, we can check that $z=-3$ is also a pole of order 1

(2) $f(z)=\frac{1-e^{-z^2}}{z^4}=\frac{1-\left[1+z^2+\frac{z^4}{2!}+\frac{z^6}{3!}+\ldots\ldots\right]}{z^4}=-\frac{1}{z^2}-\frac{1}{2!}-\frac{z^2}{3!}-\ldots\ldots$

Therefore, $z=0$ is a pole of order 2.

(3) $f(z)=\frac{z-\sin z}{z^3}=\frac{1}{z^3}\left[z-\left\{z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\ldots\right\}\right]$

$$=+\frac{1}{3!}-\frac{z^2}{5!}+\ldots\ldots$$

Therefore, $z=0$ is a removable singular point.

Example 29: Examine the nature of singularity of the functions: (a) $\sin\left(\frac{1}{1-z}\right)$, (b) $(z-3)\sin\left(\frac{1}{z+2}\right)$.

Soln: (a) $\sin\left(\frac{1}{1-z}\right)=\frac{1}{1-z}-\frac{1}{(1-z)^3.3!}+\frac{1}{(1-z)^5.5!}-\ldots\ldots$

so, $z=1$ is an isolated essential singular point.

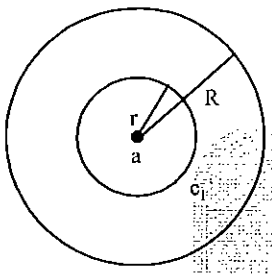
(b) $(z-3)\sin\left(\frac{1}{z+2}\right)=(z-3)\left[\frac{1}{z+2}-\frac{1}{(z+2)^3.3!}+\frac{1}{(z+2)^5.5!}-.....\right]$

so, z = -2 is an isolated essential singular point.

5.6 Residue of a Complex Function

Residue at a pole:

Let, z = a be a pole of order 'm' of f (z) and C_1 is a circle of radius 'r' with center at z = a which does not contain singularities except z = a, then f(z) is analytic within the annular region r < | z – a | <R can be expanded into Laurrent series within the annulur region as:



$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}b_n(z-a)^{-n}$$

Co-efficient b_1 is known as residue of f (z) at z = a i.e. Res. f(z = a) = $b_1=\frac{1}{2\pi i}\oint_{C_1}f(z)dz$

Methods of finding residues:

CASE 1: Residue at simple pole:

(a) Method 1: Res. f(z = a) = $\lim_{z\to a}(z-a)f(z)$

(b) Method 2: If $f(z)=\frac{\phi(z)}{\psi(z)}$ where $\psi(a)=0$ but $\phi(a)\neq 0$, then Res. f(z = a) = $\frac{\phi(a)}{\psi'(a)}$

CASE 2: Residue at a pole of order 'n':

(a) Method 1: Res. f (z = a) = $\frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=a}$

(b) Method 2: First put z - a = t and expand it into series, then Res. f (z = a) = co-efficient of 1/t.

CASE 3: Residue at z = ∞:

$$\text{Res.}f(z=\infty)=\lim_{z\to\infty}\left[-zf(z)\right]$$

Example 30: Find the singular points of the following function and the corresponding residues:

(a) $f(z)=\frac{1-2z}{z(z-1)(z-2)}$ (b) $f(z)=\frac{z^2}{z^2+a^2}$ (c) $f(z)=z^2e^{1/z}$

Soln: (a) $f(z)=\dfrac{1-2z}{z(z-1)(z-2)} \Rightarrow$ Poles : $z=0, z=1, z=2$

$$\text{Res. } f(z=0)=\underset{z\to 0}{Lt}(z-0)f(z)=\underset{z\to 0}{Lt}\frac{1-2z}{(z-1)(z-2)}=\frac{1}{2}$$

$$\text{Res. } f(z=1)=\underset{z\to 1}{Lt}(z-1)f(z)=\underset{z\to 1}{Lt}\frac{1-2z}{z(z-2)}=1$$

$$\text{Res. } f(z=2)=\underset{z\to 2}{Lt}(z-2)f(z)=\underset{z\to 2}{Lt}\frac{1-2z}{z(z-1)}=-\frac{3}{2}$$

(b) $f(z)=\dfrac{z^2}{z^2+a^2} \Rightarrow$ Poles : $z=ia, z=-ia$

$$\text{Res.} f(z=ia)=\left(\frac{z^2}{2z}\right)_{z=ia}=\frac{1}{2}ia; \text{ Res.} f(z=-ia)=\left(\frac{z^2}{2z}\right)_{z=-ia}=-\frac{1}{2}ia$$

(c) $f(z)=z^2e^{1/z}=z^2\left[1+\dfrac{1}{z}+\dfrac{1}{z^2.2!}+\dfrac{1}{z^3.3!}+.....\right] \Rightarrow$ Poles : $z=0$

$$\text{Res. } f(z=0)=\text{Coefficient of } \frac{1}{z}=\frac{1}{3!}=\frac{1}{6}$$

Example 31: The value of the residue of $\dfrac{\sin z}{z^6}$ is

(a) $-\dfrac{1}{5!}$ (b) $\dfrac{1}{5!}$ (c) $\dfrac{2\pi i}{5!}$ (d) $-\dfrac{2\pi i}{5!}$

Soln: $f(z)=\dfrac{\sin z}{z^6}=\dfrac{1}{z^6}\left[z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dfrac{z^7}{7!}+.....\right]$

$$=\frac{1}{z^5}-\frac{1}{3!}\frac{1}{z^3}+\frac{1}{5!}\frac{1}{z}-.....$$

Residue of $f(z)$ = coefficient of $\left(\dfrac{1}{z}\right)=\dfrac{1}{5!}$

Correct option is (b)

5.7 Cauchy Residue Theorem:

If f(z) in single-valued and analytic inside a simple closed curve 'C, except at a finite number of singular points within C, then

$$\oint_C f(z)dz = 2\pi i\,(\text{sum of the residues at poles within 'C'})$$

Example 32: Evaluate the integral: $\oint_C \frac{4-3z}{z(z-1)(z-3)}dz$ where $|z|=\frac{3}{2}$

Soln: $f(z)=\frac{4-3z}{z(z-1)(z-3)} \Rightarrow$ Poles : $z=0, z=1, z=3$

But, the given contour is circle centered at the origin and radius 3/2 units.

Therefore, only z = 0 and z = 1 within the contour.

$$I=2\pi i\left[\text{Re}s.f(z=0)+\text{Re}s.f(z=1)\right]=2\pi i\left[\frac{4}{3}-\frac{1}{2}\right]=\frac{5\pi i}{3}$$

Example 33: Evaluate the integral: $\oint_C \frac{e^{2z}+z^2}{(z-1)^5}dz$ *where* $|z|=2$

Soln: $f(z)=\frac{e^{2z}+z^2}{(z-1)^5} \Rightarrow$ Poles : $z=1$ (*order* 5)

$$I=2\pi i\times \text{Re}s.f(z=1)=2\pi i\times\frac{1}{4!}\frac{d^4}{dz^4}\left[e^{2z}+z^2\right]_{z=1}=2\pi i\times\frac{2e^2}{3}=\frac{4\pi ie^2}{3}$$

Example 34: The value of the integral $\oint_C \frac{z^3}{z^2-5z+6}dz$, where C is closed contour defined by the equation $2|z|-5=0$, traversed in the anti-clockwise direction, is

(a) $-16\pi i$ (b) $16\pi i$ (c) $8\pi i$ (d) $2\pi i$

Soln: $f(z)=\frac{z^3}{z^2-5z+6}$; Condition of singularity: $z^2-5z+6=0 \Rightarrow z=3,2$ (1st order pole)

Given closed contour: $c: 2|z|-5=0 \Rightarrow |z|=\frac{5}{2}$

Only z = 2 i.e. point (2, 0) is within 'c'

$$\text{Res.f}(z)\big|_{z=2}=(z-2)\frac{z^3}{(z-3)(z-2)}\bigg|_{z=2}=-8$$

Given integral: $I=2\pi i\times(\text{sum of the residues})=-16\pi i$

Correct answer is (a)

Example 35: The value of the integral $\int_C z^{10}dz$, C is the unit circle with the origin as the centre is

(a) 0 (b) $z^{11}/11$ (c) $2\pi iz^{11}/11$ (d) 1/1

Soln: $f(z)=z^{10}$ is analytic in the entire complex argand plane.

So, according to Cauchy integral theorem,

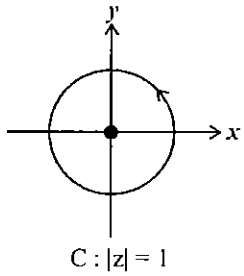
$$\int_C z^{10}dz=0$$

Correct option is (a)

Example 36: The value of the integral $\oint_C dz/z^2$, where z is a complex variable and C is the unit circle with the origin as its centre, is:

(a) 0 (b) $2\pi i$ (c) $4\pi i$ (d) $-4\pi i$

Soln: $f(z)=\frac{1}{z^2}$ has a pole of order 2 at $z=0$



Therefore, $\int_C \frac{dz}{z^2}=2\pi i$ [Residue at $z=0$]

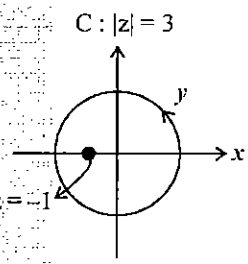
$$=2\pi i\left[\frac{1}{(2-1)!}\frac{d}{dz}\left\{z^2\cdot\frac{1}{z^2}\right\}_{z=0}\right]=0$$

Correct option is (a)

Example 37: The value of $\int_C \frac{e^{2z}}{(z+1)^4}dz$, where C is a circle defined by $|Z|=3$, is

(a) $\frac{8\pi i}{3}e^2$ (b) $\frac{8\pi i}{3}e^{-1}$ (c) $\frac{8\pi i}{3}e$ (d) $\frac{8\pi i}{3}e^2$

Soln: $f(z)=\frac{e^{2z}}{(z+1)^4}$ has a pole of order 4 at $z=-1$.



Residue of $f(z)$ at $z=-1$

$$=\frac{1}{(4-1)!}\frac{d^3}{dz^3}\left[(z+1)^4\frac{e^{2z}}{(z+1)^4}\right]_{z=-1}$$

$$=\frac{1}{3!}\left[8e^{2z}\right]_{z=-1}=\frac{4}{3}e^{-2}$$

So, $I=2\pi i\left(\frac{4}{3}e^{-2}\right)=\frac{8\pi i}{3}e^{-2}$

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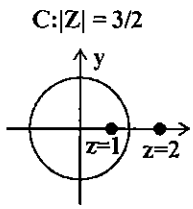
Correct option is (a)

Example 38: The value of integral $\int_C \frac{e^z}{z^2-3z+2}dz$, where the contour C is the circle $|z|=\frac{3}{2}$ is

(a) $2\pi ie$ (b) πie (c) $-2\pi ie$ (d) $-\pi ie$



Soln: $f(z)=\frac{e^z}{z^2-3z+2}$ has simple poles at $z=1$ and $z=2$

Only $z=1$ lies within the circle.

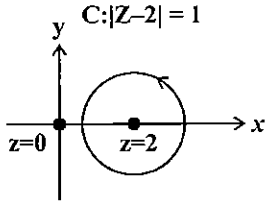
$I=2\pi i\left[\text{Residual of } f(z) \text{ at } z=1\right]$

$$=2\pi\left[\lim_{z\to 1}(z-1)\frac{e^z}{(z-1)(z-2)}\right]=2\pi i(-e)=-2\pi ie$$

Correct option is (c)

Example 39: The value of integral $\oint_C \frac{e^z \sin z}{z^2} dz$, where the contour C is the unit circle : $|z-2|=1$, is

(a) $2\pi i$ (b) $4\pi i$ (c) πi (d) 0



Soln: $f(z)=\frac{e^z \sin z}{z^2}$ has pole of order '1' at z = 0

Since, $z=0$ lies outside the circle

So, $\oint_C \frac{e^z \cdot \sin z}{z^2} dz = 0$

Correct option is (d)

Example 40: For the function $f(z)=\frac{16z}{(z+3)(z-1)^2}$, the residue at the pole $z=1$.....

(Your answer should be an integer)

Soln: $f(z)=\frac{16z}{(z+3)(z-1)^2}$ has a pole of order 2 at $z=1$

Residue of $f(z)$ at $z=1$

$$=\frac{1}{(2-1)!}\frac{d}{dz}\left[(z-1)^2\frac{16z}{(z+3)(z-1)^2}\right]_{z=1}=\frac{(z+3)\cdot 16-16z\cdot 1}{(z+3)^2}\Bigg|_{z=1}=\frac{48}{(z+3)^2}\Bigg|_{z=1}=3$$

Correct answer is (3)

Definite integrals of trigonometric functions of cosθ and sinθ :

Method:

(i) Consider the contour to be a circle centered at the origin and having radius one unit i.e. |z|=1

(ii) Assume, $z=e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$

(iii) Therefore, $\cos\theta = \frac{e^{i0}+e^{-i0}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$ and $\sin\theta = \frac{e^{i0}-e^{-i0}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right)$.

Replacements regarding $\cos\theta$ and $\sin\theta$ is to be done only in the denominator of the given integral.

(iv) The limit will be changed from $0 \to 2\pi$ *to* $\oint_C$

(v) Find the singular points and find residues at only those singular points which lies inside the unit circle.

(vi) Finally use the Cauchy residue theorem.

Example 41: Evaluate the integral: $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$; $a > b > 0$

Soln: $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{dz/iz}{a+b\left(\frac{z^2+1}{2z}\right)} = \int_0^{2\pi} \frac{2dz}{i(bz^2+2az+b)}$

The singular points are at $\Rightarrow z=\alpha=\frac{-a+\sqrt{a^2-b^2}}{b}$ & $z=\beta=\frac{-a-\sqrt{a^2-b^2}}{b}$

The singular point $z=\beta$ will lie outside the unit circle as a>b>0 while the singular point $z=\alpha$ will lie inside the unit circle which is a simple pole.

$$\text{Res.} f(z=\alpha)=\lim_{z\to\alpha}(z-\alpha)f(z)=\lim_{z\to\alpha}\frac{2}{ib}\frac{(z-\alpha)}{(z-\alpha)(z-\beta)}=\frac{2}{ib(\alpha-\beta)}=\frac{2}{ib}\times\frac{b}{2\sqrt{a^2-b^2}}=\frac{1}{i\sqrt{a^2-b^2}}$$

Therefore, by cauchy Residue theorem.

$$I=2\pi i\times\text{Residue}=2\pi i\times\frac{1}{i\sqrt{a^2-b^2}}=\frac{2\pi}{\sqrt{a^2-b^2}}$$

Example 42: Evaluate the integral $\int_0^{2\pi}\frac{d\theta}{5+4\cos\theta}$

Soln: $\int_0^{2\pi}\frac{d\theta}{5+4\cos\theta}=\oint_C\frac{1}{5+4.\frac{1}{2}\left(z+\frac{1}{z}\right)}.\frac{dz}{iz}=\frac{1}{i}\oint_C\frac{dz}{\left(5z+2z^2+2\right)}=\oint_C\frac{dz}{(2z+1)(z+2)}$

$f(z)=\frac{1}{(2z+1)(z+2)}$ $\Rightarrow$ poles are at $z=-\frac{1}{2},-2$ (Only $z=-½$ is within the circle)

$$\text{Res. f}(z=-½)=\left(z+\frac{1}{2}\right).\left.\frac{1}{(2z+1)(z+2)}\right|_{z=-\frac{1}{2}}=\left.\frac{1}{2(z+2)}\right|_{z=-\frac{1}{2}}=\frac{1}{3}$$

Hence, the integral $=\frac{2\pi i}{3}$

Example 43: Evaluate the integral $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta-n\theta)d\theta$

Soln: $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta-n\theta)d\theta$ = Real part of $\int_0^{2\pi} e^{\cos 0}e^{i(\sin\theta-n\theta)}d\theta$

$$= \text{Real part of}\int_0^{2\pi} e^{(\cos\theta+i\sin\theta)}e^{-in\theta}d\theta = \text{Real part of}\oint_C e^z.z^{-n}\frac{dz}{iz}$$

$$= \text{Real part of}\frac{1}{i}\oint_C\frac{e^z}{z^{n+1}}dz$$

$f(z)=\frac{e^z}{z^{n+1}}$ $\Rightarrow$ poles are at $z=0$ of order n $\Rightarrow$ Res. $f(z=0)=\frac{1}{n!}.\frac{d^n}{dz^n}\left(z^{n+1}.\frac{e^z}{z^{n+1}}\right)\Big|_{z=0}=\frac{1}{n!}$

Hence the given integral = real part of $\left(\frac{1}{i}.2\pi i.\frac{1}{n!}\right)=\frac{2\pi}{n!}$

Evaluation of real improper integrals between the limit $-\infty$ to $+\infty$:

Case 1:

$$f(x)=\frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are alzebraic functions of x.

First assume $f(z)=\frac{p(z)}{q(z)}$ and find the singular points of $f(z)$.

$$\int_{-\infty}^{\infty} f(x)\,dx=\int_{-\infty}^{\infty} f(z)\,dz=2\pi i\left[\Sigma \text{ Res. at all singular points in the upper half of the complex plane}\right]$$
$$+\pi i\left[\Sigma \text{ Res. at all singular points on the real axis}\right]$$

Example 44: Calculate the following integral $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$

Soln: $f(x)=\frac{1}{1+x^2}$; Assume $f(z)=\frac{1}{1+z^2}$

$f(z)=\frac{1}{1+z^2}$ has simple poles at $z=\pm i$.

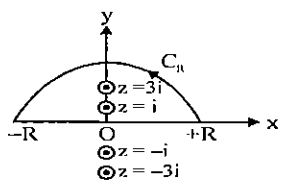
Here, $z=i$ lies in the upper half of the complex plane.

So, $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\int_{-\infty}^{\infty}\frac{dz}{1+z^2}=2\pi i\left[\text{Res. } f(z=i)\right]=2\pi i\left[\frac{1}{2i}\right]=\pi$

Example 45: Apply calculus of residues to show that $\int_0^{\infty}\frac{dx}{(x^2+1)(x^2+9)}=\frac{\pi}{24}$

Soln: $I=\int_0^{\infty}\frac{dx}{(x^2+1)(x^2+9)}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+9)}$

Let $f(z)=\frac{1}{(z^2+1)(z^2+9)}$



The singular points of f(z) are

$(z^2+1)(z^2+9)=0 \quad \Rightarrow \quad z=i,-i,3i,-3i$ (Simple poles)

The only poles lying in the upper half plane are at z = +i and +3i

$$\int_{-\infty}^{\infty} f(z)dz = \int_{-\infty}^{\infty} f(x)dx = 2\pi i\left(\text{Res. at } (z=i) + \text{Res. at } (z=3i)\right)$$

Therefore, $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)} = 2\pi i\left[\frac{1}{16i} - \frac{1}{48i}\right] = \frac{\pi}{12} \Rightarrow \int_{0}^{\infty} \frac{dx}{(x^2+1)(x^2+9)} = \frac{\pi}{24}$

Example 46: The value of the integral $\int_{-\infty}^{\infty} \frac{1}{1+x^4}dx$ is

(a) $\pi/\sqrt{2}$ (b) $\pi/2$ (c) $\sqrt{2}\pi$ (d) 2π

Soln: Assume, $f(z) = \frac{1}{1+z^4}$

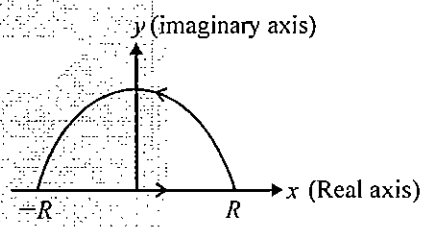
Condition of singularity : $1+z^4 = 0 \Rightarrow z^4 = -1 = e^{i(2\pi+1)\pi} \Rightarrow z = e^{i(2n+1)\pi/4}$ $(n = 0, 1, 2, 3)$

Poles of $f(z)$ will be $z = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$ and will be order 1.

Out of them, $z = e^{i\pi/4}, e^{3i\pi/4}$ lies in the upper half of complex plane.

$$\text{Res} \cdot f\left(z = e^{i\pi/4}\right) = \frac{1}{4z^3} = \frac{1}{4e^{i3\pi/4}}$$

$$\text{Res} \cdot f\left(z = e^{i3\pi/4}\right) = \frac{1}{4z^3} = \frac{1}{4e^{i9\pi/4}}$$



$$\therefore \quad \int_{-\infty}^{\infty} \frac{1}{1+x^4}dx = 2\pi i \frac{1}{4}\left[\frac{1}{e^{3i\pi/4}} + \frac{1}{e^{i9\pi/4}}\right]$$

$$= \frac{2\pi i}{4}\left[e^{-3i\pi/4} + e^{-i9\pi/4}\right] = \frac{2\pi i}{4}\left[\frac{1}{\sqrt{2}}(-1-i) + \frac{1}{\sqrt{2}}(1-i)\right] = \frac{4\pi}{4\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

Correct option is (a)

CAREER ENDEAVOUR

Case 2:

$$f(x) = \frac{p(x)}{q(x)}\cos mx \text{ or } \frac{p(x)}{q(x)}\sin mx$$

where $p(x)$ and $q(x)$ are alzebraic functions of x.

First assume $f(z) = \frac{p(z)}{q(z)}e^{imz}$ and find the singular points of $f(z)$.

$$\int_{-\infty}^{\infty} f(x)dx = \text{R.P. or I.P. of } \int_{-\infty}^{\infty} f(z)dz$$

$$= \text{R.P. or I.P. of } \{2\pi i[\Sigma \text{ Res. at all singular points in the upper half of the complex plane}]$$
$$+ \pi i[\Sigma \text{ Res. at all singular points on the real axis}]\}$$

Example 47: Prove $\int_{-\infty}^{\infty} \frac{\cos x dx}{\left(x^2+a^2\right)} = \frac{\pi}{a} e^{-a} \quad ; a > 0$

Soln: Let $f(z) = \frac{e^{iz}}{\left(z^2+a^2\right)}$; Condition of singularity: $z^2 + a^2 = 0 \Rightarrow z = \pm ia$.

The only pole which lies in the upper half of the circle is at z = ia.

Therefore, Residue at (z = ia) = $\lim_{z \to ia} \frac{(z-ia)\, e^{iz}}{(z-ia)(z+ai)} = \frac{e^{-a}}{2ia}$

So, $\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+a^2} dz = 2\pi i \left[\text{Residue at } z = ia\right] = 2\pi i \left[\frac{e^{-a}}{2ia}\right] = \frac{\pi}{a} e^{-a}$

Hence, $\int_{-\infty}^{\infty} \frac{\cos x}{\left(x^2+a^2\right)} dx$ = R.P. of $\int_{-\infty}^{\infty} \frac{e^{ix}}{\left(x^2+a^2\right)} dx = \frac{\pi}{a} e^{-a} \Rightarrow \int_0^{\infty} \frac{\cos x}{\left(x^2+a^2\right)} dx = \frac{\pi}{2a} e^{-a}$

Example 48: If $a > 0$, $m > 0$, then

$$\int_0^{\infty} \frac{\cos mx dx}{\left(x^2+a^2\right)^2} = \frac{\pi}{4a^3}(1+ma)e^{-ma}$$

Soln: $f(z) = \frac{e^{imz}}{\left(a^2+z^2\right)^2}$ has poles of order 2 at $z = \pm ia$

Only $z = ia$ is in the upper half of the complex plane.

$$\int_{-\infty}^{\infty} \frac{e^{imz}}{\left(z^2+a^2\right)^2} dx = 2\pi i \left[\text{Res}(z = ia)\right] = 2\pi i e^{-ma} \frac{(1+ma)}{4a^3 i} = \frac{\pi}{2a^3} e^{-ma}(1+ma)$$

CAREER ENDEAVOUR

Equating real parts from both sides,

$$\int_{-\infty}^{\infty} \frac{\cos mx dx}{\left(x^2+a^2\right)^2} = \frac{\pi}{2a^3}(1+ma)e^{-ma}$$

But $\int_{-\infty}^{\infty} \frac{\cos mx dx}{\left(x^2+a^2\right)^2} dx = 2\int_0^{\infty} \frac{\cos mx}{\left(x^2+a^2\right)^2} dx$

Hence, the last gives $\int_0^{\infty} \frac{\cos mx}{\left(x^2+a^2\right)^2} dx = \frac{\pi}{4a^3}(1+ma)e^{-ma}$

Example 49: The principal value of the integral $\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3} dx$ is

(a) -2π (b) $-\pi$ (c) π (d) 2π

Soln: Given integral: $\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3} dx$ = Imaginary part of $\int_{-\infty}^{\infty} \frac{e^{i2x}}{x^3} dx$

Assume, the complex function to be $f(z) = \frac{e^{i2z}}{z^3}$

Singular points of $f(z)$: $z = 0$ is pole of order 3
Since, the pole lie on the real axis, therefore,

$$\int_{-\infty}^{\infty} \frac{e^{i2z}}{z^3} dz = \pi i \left[\text{Residue of } f(z) \text{ at } z = 0\right]$$

$$= \pi i \left[\frac{1}{(3-1)!} \frac{d^2}{dz^2} \left\{(z-0)^3 \frac{e^{i2z}}{z^3}\right\}\right]_{z=0}$$

$$= \pi i \left[\frac{1}{2}\left(4i^2\right) e^{i2z}\right]_{z=0} = -2\pi i$$

So, the required integral $\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3} dx$ = Imaginary part of $(-2\pi i) = -2\pi$

Correct Option is (a)

TIFR PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. If $z = x + iy$ then the function

$$f(x,y) = (1+x+y)(1+x-y) + a(x^2 - y^2) - 1 + 2iy(1-x-ax)$$

where a is a real parameter, is analytic in the complex z plane if a is equal to

(a) -1 (b) $+1$ (c) 0 (d) i

Soln: $f(x,y) = (1+x+y)(1+x-y) + a(x^2 - y^2) - 1 + 2iy(1-x-ax)$

Comparing this with, $f(x,y) = u(x,y) + iv(x,y)$

We have, $u = (1+x+y)(1+x-y) + a(x^2 - y^2) - 1$ and $v = 2y(1-x-ax)$

For $f(x,y)$ to be analytic, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow (1+x+y) + (1+x-y) + 2ax = 2(1-x-ax)$

$\Rightarrow 2 + 2x + 2ax = 2 - 2x - 2ax$

$\Rightarrow 4x + 4ax = 0 \Rightarrow 4x(1+a) = 0$

Since, $x \neq 0 \Rightarrow 1 + a = 0$ or $a = -1$.

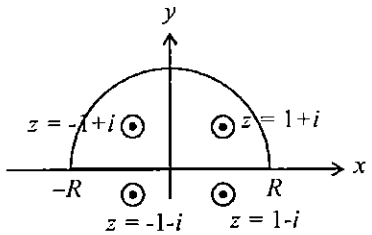
Correct option is (a)

Question 2. The integral $\int_0^{\infty} \frac{dx}{4+x^4}$ evalautes to **[TIFR 2014, 2017]**

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

Soln: We have, $\int_0^{\infty} \frac{dx}{4+x^4}$

Let $f(z)=\frac{1}{z^4+4}$



The singular points of $f(z)$ are

$z^4 = -4 = 4(\cos\pi + i\sin\pi) = 4(\cos(2n+1)\pi + i\sin(2n+1)\pi)$

$\Rightarrow \quad z = \sqrt{2}\left[\cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}\right] \qquad n = 0,1,2,3$

$z_1 = (1+i), z_2 = (-1+i), z_3 = (-1-i), z_4 = (1-i)$ out of this four poles only $(1+i)$ and $(-1+i)$ lie with in the contour.

Now, $\int_0^{\infty} \frac{dz}{z^4+4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz}{z^4+4} = \pi i$ (sum of the residues with in contour)

$$= \pi i\left[\text{Residue at } z = \sqrt{2}e^{\frac{i\pi}{4}} + \text{Residue at } z = \sqrt{2}e^{\frac{3i\pi}{4}}\right]$$

$$= \pi i\left[\frac{1}{8\sqrt{2}}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) + \frac{1}{8\sqrt{2}}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi i}{16}[-1-i+1-i] = \frac{\pi}{8}$$

CAREER ENDEAVOUR

Correct option is (d)

Question 3. The integral $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$ where $0 < a < 1$, evaluates to **[TIFR 2015]**

(a) $\frac{2\pi}{1-a^2}$ (b) $\frac{2\pi}{1+a^2}$ (c) 2π (d) $\frac{4\pi}{1-a^2}$

Soln: $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2} = \int_0^{2\pi} \frac{d\theta}{1-2a\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)+a^2} = \int_0^{2\pi} \frac{d\theta}{1-a\left(e^{i\theta}+e^{-i\theta}\right)+a^2}$

Let $e^{i\theta} = z$ and C is the unit circle $|z| = 1$

Therefore, $ie^{i\theta}\,d\theta = dz \Rightarrow d\theta = \frac{dz}{iz}$

Therefore, $\int_0^{2\pi} \frac{d\theta}{1-a\left(e^{i\theta}+e^{-i\theta}\right)+a^2} = \int_C \frac{dz}{\left[1-a\left(z+\frac{1}{z}\right)+a^2\right] iz}$

$$= \frac{1}{i}\int_C \frac{dz}{z-az^2-a+za^2} = \frac{1}{i}\int_C \frac{dz}{(z-a)(1-az)}$$

Poles are at $z=a, \frac{1}{a}$, since $a<1$, only $z=a$ lies within the contour.

Residue at $z=a$, $\lim_{z\to a}\left((z-a)\frac{1}{(z-a)(1-az)}\right) = \frac{1}{1-a^2}$

Therefore, value of integral $= \frac{2\pi i}{i}$ (Sum of residue)

$$= 2\pi\left(\frac{1}{1-a^2}\right)$$

$$= \frac{2\pi}{1-a^2}.$$

Correct option is (a)

Question 4. The value of the integral $\oint_C \frac{\sin z}{z^6} dz$, where C is the circle with centre $z = 0$ and radius 1 unit **[TIFR 2016]**

(a) $i\pi$ (b) $\frac{i\pi}{120}$ (c) $\frac{i\pi}{60}$ (d) $-\frac{i\pi}{6}$

Soln: $f(z) = \frac{\sin z}{z^6}$ has a pole at $z = 0$.

$$\frac{\sin z}{z^6} = \frac{1}{z^6}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)$$

CAREER ENDEAVOUR

Residue at $(z=0)$ = co-efficient of $1/z = \frac{1}{5!} = \frac{1}{120}$

Therefore, $\oint_C \frac{\sin z}{z^6} dz = 2\pi i \times \frac{1}{120} = \frac{\pi i}{60}$.

Correct option is (c)

Question 5. The value of the integral $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx$ is

(a) $\frac{1}{2a}$ (b) $\frac{1}{2\pi a}$ (c) $\pi a \exp(-a)$ (d) $\exp(-a)/a$

Soln: $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx$ = Real part of $\frac{1}{\pi}\int_C \frac{e^{iz}}{z^2+a^2} dz$,

Taken round the closed contour C, consisting of the upper half of a large circle $|z| = R$ and the real axis from $-R$ to R.

Poles are at $z^2 + a^2 = 0$, $\boxed{z = \pm ia}$ and Only $z = ia$ lies inside the contour

Residue at $(z = ia) = \lim_{z \to ia}\left(\frac{e^{iz}(z - ia)}{(z + ia)(z - ia)}\right) = \frac{e^{ixia}}{ia + ia} = \frac{e^{-a}}{2ia}$

Therefore, value of integral $= \frac{1}{\pi} 2\pi i \times \frac{e^{-a}}{2ia} = \frac{e^{-a}}{a}$.

Correct option is (d)

JEST PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The value of integral $\int_0^\infty \frac{\ln x}{(x^2+1)^2} dx$ is **[JEST 2012]**

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

Soln: $I = \int_0^\infty \frac{\ln x}{(x^2+1)^2} dx$

Let $f(z) = \frac{\ln z}{(z^2+1)^2}$

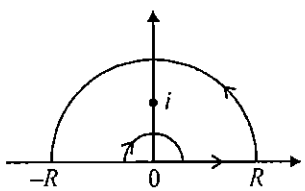
Singular points are $z = 0$ (pole of order 1), and $z = \pm i$ (poles of order 2).

Hence, residue at $z = 0$

$$\lim_{z \to 0}(z - 0)\frac{\ln(z)}{(z^2+1)^2} \text{ replace } z \text{ by } \frac{1}{t}$$

Therefore, $\lim_{t \to 0} \frac{1}{t} \frac{\ln\left(\frac{1}{t}\right)}{\left(\frac{1}{t^2}+1\right)^2} \Rightarrow \lim_{t \to 0} -\frac{\ln(t)}{t} \times \frac{1}{\left(\frac{1}{t^2}+1\right)^2} = 0$ (Using L-Hospital's Rule)

Residue at $z = i$ (only $z = i$ lies within upper half)



$$\lim_{z \to i} \frac{d}{dz}\left[\frac{(z-i)^2 \ln(z)}{(z+i)^2(z-1)^2}\right]$$

$$\lim_{z \to i} \frac{d}{dz}\left[\frac{\ln(z)}{(z+i)^2}\right] \Rightarrow \lim_{z \to i}\left[\frac{(z+i)^2 \times \frac{1}{z} - (\ln z)2(z+i)}{(z+i)^4}\right]$$

$$\Rightarrow \lim_{z \to i}\left[\frac{(z+i)\frac{1}{z} - 2\ln z}{(z+i)^3}\right] = \frac{2i \times \frac{1}{i} - 2\ln(i)}{(2i)^3} = \frac{2 - 2\ln(e^{i\pi/2})}{-8i} = \frac{2 - i\pi}{-8i}$$

Hence, according to Cauchy residue theorem,

$$\int_{-\infty}^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx = \pi i \text{ (residue at } z=0) + 2\pi i \text{ (residue at } z=i)$$

$$= \pi i \times 0 + 2\pi i \times \left(\frac{2-i\pi}{-8i}\right) = \frac{2\pi}{-8} \times 2 + \frac{2\pi}{8} i\pi = -\frac{\pi}{2} + i\frac{\pi^2}{4}.$$

or $$\int_{-\infty}^{0} \frac{\ln x}{(x^2+1)^2}\,dx + \int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx = -\frac{\pi}{2} + i\frac{\pi^2}{4} \quad \ldots (1)$$

Now, $\int_{-\infty}^{0} \frac{\ln x}{(x^2+1)^2}\,dx$ (Replacing x by $-x$)

$$= \int_{\infty}^{0} \frac{\ln(-x)}{(x^2+1)^2}(dx) = -\int_{\infty}^{0} \frac{\ln(e^{i\pi}x)}{(x^2+1)^2}\,dx$$

$$= i\pi \int_{0}^{\infty} \frac{dx}{(x^2+1)^2}\,dx + \int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx$$

$$= i\pi \times \frac{\pi}{4} + \int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx$$

Therefore, from Eq. (1);

$$i\frac{\pi^2}{4} + \int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx = -\frac{\pi}{2} + i\frac{\pi^2}{4} \text{ or } \int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx = -\frac{\pi}{2}.$$

Correct option is (c)

Question 2. Compute $\lim_{z\to 0} \frac{\mathrm{Re}(z^2)+\mathrm{Im}(z^2)}{z^2}$ **[JEST 2013]**

(a) the limit does not exist (b) 1

(c) –i (d) –1

Soln: Let $z = x + iy$, then $z^2 = x^2 - y^2 + 2ixy$

Therefore, $\mathrm{Re}(z^2) = x^2 - y^2$, $\mathrm{Im}(z^2) = 2xy$

Now, $$\lim_{z\to 0} \frac{\mathrm{Re}(z^2)+\mathrm{Im}(z^2)}{z^2} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{(x^2-y^2)+2xy}{(x^2-y^2)+i2xy}$$

Let $y = mx$, then $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{(x^2-m^2x^2)+2mx^2}{(x^2-m^2x^2)+2imx^2} = \frac{1-m^2+2m}{1-m^2+2im}$$

Since, this depends on m, and hence on the path along which $x \to 0$, $y \to 0$.

Hence, limit does not exists.

Correct option is (a)

Question 3. The value of integral $I=\oint_c \frac{\sin z}{2z-\pi}dz$

with c a is circle $|z|=2$, is **[JEST 2014]**

(a) 0 (b) $2\pi i$ (c) πi (d) $-\pi i$

Soln: $\frac{\sin z}{2z-\pi}=\frac{\sin z}{2\left(z-\frac{\pi}{2}\right)}=\frac{\cos\left(z-\frac{\pi}{2}\right)}{2\left(z-\frac{\pi}{2}\right)}=\frac{\cos t}{2t}$ (Let $z-\frac{\pi}{2}=t$)

$$=\frac{1}{2t}\left[1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+.....\right]=\frac{1}{2}\frac{1}{t}-\frac{t}{2\times 2!}+\frac{t^3}{2\times 4!}-\frac{t^5}{2\times 6!}$$

The residue at $t=0$ is the coefficient of $\frac{1}{z}$ i.e. $\frac{1}{2}$

Therefore, the pole $z=\frac{\pi}{2}$ lies within the contour C and the residue at $z=\frac{\pi}{2}$ is $\frac{1}{2}$

$\therefore \quad \oint_C \frac{\sin z}{2z-\pi}dz=2\pi i\left[\text{sum of residue}\right]=2\pi i\times\frac{1}{2}=\pi i$

Correct option is (c)

Question 4. The value of limit $\lim_{z\to i}\frac{z^{10}+1}{z^6+1}$ is equal to **[JEST 2014]**

(a) 1 (b) 0 (c) $-10/3$ (d) $5/3$

Soln: $\lim_{z\to i}\frac{z^{10}+1}{z^6+1}$, Here $(i)^{10}=(i^2)^5=-1$; $i^6=(i^2)^3=-1$

Hence, given that limit is of 0/0 form.

Therefore, using L-Hospital's rule,

$$\lim_{z\to i}\frac{10z^9}{6z^5}\Rightarrow\frac{10(i)^9}{6(i)^5}=\frac{5}{3}\frac{i}{i}=\frac{5}{3}.$$

Correct option is (d)

Question 5. Given an analytic function $f(x,y)=\phi(x,y)+i\psi(x,y)$ where **[JEST 2015]**

$\phi(x,y)=x^2+4x-y^2+2y$. If C is a constant, then which of the following relations is true?

(a) $\psi(x,y)=x^2y+4y+C$ (b) $\psi(x,y)=2xy-2x+C$

(c) $\psi(x,y)=2xy+4y-2x+C$ (d) $\psi(x,y)=x^2y-2x+C$

Soln: The condition of analytic function are

$$\frac{\partial \phi}{\partial y}=\frac{\partial \psi}{\partial x} \qquad \left| \qquad \frac{\partial \phi}{\partial y}=-\frac{\partial \psi}{\partial x}\right.$$

We have, $\phi(x,y)=x^2+4x-y^2+2y$

Therefore, $\frac{\partial \phi}{\partial x}=2x+4$ And $\frac{\partial \phi}{\partial y}=-2y+2$

$\Rightarrow \quad \frac{\partial \psi}{\partial y}=2x+4 \qquad \Rightarrow \quad -\frac{\partial \psi}{\partial x}=-2y+2$

$\Rightarrow \quad \psi=2xy+4y+c_1(x) \qquad \Rightarrow \quad \psi=2xy-2x+c_1(y)$

Therefore, $\psi(x,y)=2xy+4y-2x+C$ (Numerical constant)

Correct option is (c)

Question 6. The value of the integral $\int_0^\infty \frac{\ln x}{(x^2+1)}dx$ is **[JEST 2016]**

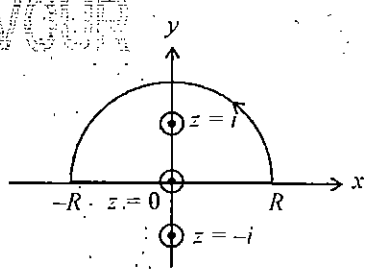
(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{2}$ (c) π^2 (d) 0

Soln: $\int_0^\infty \frac{\ell nx}{(x^2+1)}dx$

Say, $f(z)=\frac{\ell nz}{(z^2+1)}$ Singular points : $z=0,\pm i \Rightarrow$ pole of order 1

$$\text{Res}\cdot f(z=0)=\lim_{z\to 0}(z-0)\frac{\ell nz}{(z^2+1)}$$

$$=\lim_{z\to\infty}\left(\frac{1}{t}\right)\frac{\ell n\left(\frac{1}{t}\right)}{(1/t^2+1)} \quad [\text{Replacing } z \text{ by } \frac{1}{t}]$$



$$=\lim_{t\to\infty}\left(-\frac{\ell nt}{t}\right)\frac{1}{(1/t^2+1)}=0 \qquad \text{(Using L' Hospital's rule)}$$

$$\text{Res}\cdot f(z=i)=\lim_{z\to i}(z-i)\frac{\ell nz}{(z^2+1)}=\lim_{z\to i}\frac{nz}{(z+i)}=\frac{\ell n(i)}{2i}=\frac{\ell n\left(e^{i\pi/2}\right)}{2i}=\frac{i\pi/2}{2i}=\frac{\pi}{4}$$

According to Cauchy's Residue Theorem,

$$\int_{-\infty}^{\infty}\frac{\ell nx}{(x^2+1)}dx=\pi i\left[\text{Res}\cdot f(z=0)\right]+2\pi i\left[\text{Res}\cdot f(z=i)\right]=i\pi^2/2$$

$$\Rightarrow \quad \int_{-\infty}^{0} \frac{\ell nx}{\left(x^2+1\right)} dx + \int_{0}^{\infty} \frac{\ell nx}{\left(x^2+1\right)} dx = \frac{i\pi^2}{2} \qquad \ldots (i)$$

Now, $\int_{-\infty}^{0} \frac{\ell nx}{\left(x^2+1\right)} dx$ (Replacing x by $-x$)

$$= \int_{\infty}^{0} \frac{\ell n(-x)}{\left(x^2+1\right)}(-dx) = -\int_{\infty}^{0} \frac{\ell n\left(e^{i\pi}x\right)}{\left(x^2+1\right)} dx$$

$$= i\pi \int_{0}^{\infty} \frac{dx}{\left(x^2+1\right)} + \int_{0}^{\infty} \frac{\ell nx}{\left(x^2+1\right)} dx \qquad \ldots (ii)$$

Using (ii) in equation (i), we get

$$i\pi \int_{0}^{\infty} \frac{dx}{\left(x^2+1\right)} + 2\int \frac{\ell nx}{\left(x^2+1\right)} dx = \frac{i\pi^2}{2}$$

Comparing real parts, $\int_{0}^{\infty} \frac{\ell nx}{\left(x^2+1\right)} dx = 0$

Correct option is (d)

Question 7. The integral $I = \int_{1}^{\infty} \frac{\sqrt{x-1}}{(1+x)^2} dx$ is **[JEST 2017]**

(a) $\frac{\pi}{\sqrt{2}}$ (b) $\frac{\pi}{2\sqrt{2}}$ (c) $\frac{\sqrt{\pi}}{2}$ (d) $\sqrt{\frac{\pi}{2}}$

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Soln: $I = \int_{1}^{\infty} \frac{\sqrt{x-1}}{(1+x)^2} dx$

Let $x-1 = e^{2i\theta} \quad \Rightarrow \quad dx = e^{2i\theta} \times 2id\theta$

and $x+1 = e^{2i\theta} + 2$, as $x \to 1 \quad \theta \to -\infty$

and as $x \to \infty \quad \theta \to \infty$

$$\int_{-\infty}^{\infty} \frac{e^{i\theta}}{(e^{2i\theta}+2)^2} 2ie^{2i\theta} d\theta$$

Let $z = e^{i\theta} id\theta \Rightarrow dz = e^{i\theta} id\theta$

Therefore, $\int_{-\infty}^{\infty} \frac{2z^2 dz}{(z^2+2)^2}$

Singular points are $z^2 + 2 = 0 \Rightarrow z = \pm\sqrt{2}i$.

Only $z = \pm\sqrt{2}i$ lies in the upper half.

Hence Residue at $z = \sqrt{2}i$

$$= \lim_{z\to\sqrt{2}i} \frac{d}{dz}\left((z-\sqrt{2}i)^2 \frac{2z^2}{(z+\sqrt{2}i)^2(z-2i)^2}\right) = \lim_{z\to\sqrt{2}i} \frac{d}{dz}\left(\frac{2z^2}{(z+\sqrt{2}i)^2}\right)$$

$$= \lim_{z\to\sqrt{2}i} \left(\frac{(z+\sqrt{2}i)^2 \times 2z - 2z^2 2(z+\sqrt{2}i)}{(z+\sqrt{2}i)^4}\right) = \lim_{z\to\sqrt{2}i} \left(\frac{(z+\sqrt{2}i)4z - 4z^2}{(z+\sqrt{2}i)^3}\right)$$

$$= \lim_{z\to\sqrt{2}i} \frac{4\sqrt{2}iz}{(z+\sqrt{2}i)^3} = \frac{i4\sqrt{2}\times\sqrt{2}i}{(2\sqrt{2}i)^3} = \frac{-8}{-8\times 2\sqrt{2}i} = \frac{-i}{2\sqrt{2}}$$

Hence, $I = 2\pi i \times \frac{-i}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$

Correct option is (a)

JNU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The function $\frac{1}{\cosh x}$ may be expressed around the point $x = 0$ as a power series as **[JNU 2012]**

(a) $1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+....$ (b) $1-\frac{1}{2}x^2+\frac{5}{24}x^4-\frac{61}{720}x^6+....$

(c) $1-\frac{1}{2}x^2+\frac{11}{24}x^4-\frac{331}{720}x^6+....$ (d) $1-\frac{1}{2x^2}+\frac{1}{24x^4}-\frac{1}{720x^6}+....$

Soln: $S = \frac{1}{\cosh x} = (\cosh x)^{-1} = \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)^{-1}$

$$= 1+\left(\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)+\left(\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)^2$$

$$= 1-\frac{x^2}{2}+\left(-\frac{x^4}{4!}+\frac{x^4}{(2)^2}\right)+\left(-\frac{x^6}{6!}+\frac{x^2}{2!}\frac{x^4}{4!}\right)+\cdots$$

$$= 1-\frac{x^2}{2}+\frac{5}{24}x^4-\frac{61}{720}x^6+\cdots$$

Correct option is (b)

Question 2. The contour integral

$$\oint_C \frac{zdz}{(z-1)^2}$$

where the contour C is a circle of radius 2, is obtained as **[JNU 2013]**

(a) $2\pi i$ (b) 0 (c) πi (d) $4\pi i$

Soln: $I = \oint \frac{z\,dz}{(z-1)^2}$

Poles are at $z=1$ of second order.

Hence residue at $(z=1)=\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2\frac{z}{(z-1)^2}\right)=\lim_{z\to 1}(1)=1.$

Therefore, $I=2\pi i$ (residue at $z=1$)

$=2\pi i.$

Correct option is (a)

Question 3. What is the residue of the complex function $f(z)=\exp(1/z)$ at $z=0$? **[JNU 2017]**

(a) 1 (b) 0 (c) Undefined (d) e (e) ∞

Soln: $f(z)=\exp\left(\frac{1}{z}\right)=1+\frac{1}{z}+\frac{1}{2!z^2}+\cdots$

Therefore, residue at $z=0$, co-efficient of $\frac{1}{z}=1.$

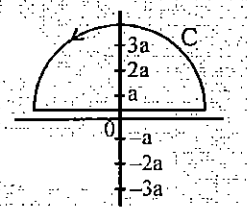
Correct option is (a)

DU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. Consider the integral of the function f(z) in the complex plane along the contour C as shown in the figure. For $f=[z^2+a^2]^{-1}$, the value of the integral is: **[DU 2014]**

(a) $\frac{\pi}{a}$ (b) $\frac{2\pi}{a}$

(c) $\frac{i\pi}{2a}$ (d) $\frac{i\pi}{a}$



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Soln: $I=\oint_C f(z)\,dz=\oint_C\frac{1}{z^2+a^2}dz$

Poles are at $z^2+a^2=0\Rightarrow z=\pm ia$

Only $z=ia$ lies within the given contour

Therefore, residue at $z=ia$

$$R=\lim_{z\to ia}\left((z-ia)\times\frac{1}{(z+ia)(z-ia)}\right)\Rightarrow R=\frac{1}{2ia}$$

Therefore, $I=2\pi i\times R=2\pi i\times\frac{1}{2ia}=\frac{\pi}{a}.$

Correct option is (a)

Question 2. The function $f(z)=\frac{(z-2)}{(z^2-4)(z+3)}$ **[DU 2014]**

(a) Has a simple pole at z = 2 and a simple pole at z = 3

(b) Has a simple pole at z = –2 and a simple pole at z = –3

(c) Has a pole of order 2 at z = 2 and a simple pole at z = –3

(d) Has a pole of order 2 at z = –2 and a simple pole at z = 3

Soln: $f(z)=\dfrac{z-2}{(z^2-4)(z+3)}=\dfrac{z-2}{(z-2)(z+2)(z+3)}$

Hence, $f(z)$ has poles at $z=2,-2,-3$, which are simple.

Correct option is (b)

Question 3. The complex function $f(z) = z^{1/2}$ **[DU 2014]**

(a) Has a simple pole at z = 0

(b) Has a pole of order 2 at z = 0

(c) Has a branch point

(d) Is analytic in the entire complex plane

Soln: Correct option is (c)

Question 4. The imaginary part of an analytic function in Cartesian coordinate is $2xy$. The real part is given by

(a) $x^2+y^2=a^2$

(b) $x^2-y^2=a^2$ **[DU 2015]**

(c) $-x^2+y^2=a^2$

(d) $-x^2-y^2=a^2$

Soln: Let $f(z)=u(x,y)+v(x,y)$ and $z=x+iy$

Given : $v(x,y)=2xy$

$$v_x=\frac{\partial v}{\partial x}=2y\Rightarrow v_x(z,0)=0 \quad \text{and} \quad v_y=\frac{\partial v}{\partial y}=2x\Rightarrow v_y(z,0)=2z$$

Therefore, $f(z)=\int\left(v_y(z,0)+iv_x(z,0)\right)dz+c$

$$=\int(2z\,dz+0)\,dz+c$$

$$=2\frac{z^2}{2}+c$$

$$f(z)=z^2=(x^2-y^2+2ixy)$$

CAREER ENDEAVOUR

Hence, real part is $\left(x^2-y^2\right)$.

Correct option is (b)

Question 5. The Laurent series of a function $f(z)$ about $z=2$ is given by **[DU 2015]**

$$f(z)=\frac{1}{2(z-2)^3}-\frac{1}{4(z-2)^2}+\frac{1}{8(z-2)}-\frac{1}{16}+\frac{z-2}{32}-\ldots..$$

The order of the pole and the value of the residue are

(a) Order = 3 and Residue = $\dfrac{1}{2}$

(b) Order = 2 and Residue = $-\dfrac{1}{4}$

(c) Order = 1 and Residue = $\dfrac{1}{8}$

(d) Order = 3 and Residue = $\dfrac{1}{8}$

Soln: Laurent series of $f(z)$ about $z=2$

$$f(z)=\frac{1}{2(z-2)^3}-\frac{1}{4(z-2)^2}+\frac{1}{8(z-2)}-\frac{1}{16}+\frac{z-2}{3^2}\cdots$$

From this expansion, order = 3 (Highest negative power) and residue at $z=2$ is 1/8 (Co-efficient of $1/z-2$).

Correct option is (d)

Question 6. In which quadrants of the complex plane, is the function $f(z)=|x|-i|y|$ analytic? **[DU 2015]**

(a) Only in the fourth quadrant
(b) Only in the first and third quadrants
(c) Only in the second and fourth quadrants
(d) In all the quadrants

Soln: In 4th quadrant $f(z)=x+iy$.

Hence, $u=x,\ v=y$; $\frac{\partial u}{\partial x}=1,\ \frac{\partial v}{\partial y}=1$, and $\frac{\partial u}{\partial y}=0=\frac{\partial v}{\partial y}$

In 2nd quadrant $f(z)=-x-iy$

Hence, $u=-x,\ v=-y$; $\frac{\partial u}{\partial x}=-1,\ \frac{\partial v}{\partial y}=-1$, and $\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=0$

Correct option is (c)

Question 7. If $z=x+iy$, then the real part of $\cos(z)$ is: **[DU 2017]**

(a) $\cos(x)\cosh(y)$ (b) $\cos(x)\cos(y)$ (c) $\sin(x)\sinh(y)$ (d) $-\sin(x)\sinh(y)$

Soln: $\cos z=\cos(x+iy)\ =\cos x\cos iy-\sin x\sin iy\ =\cos x\cosh y-i\sin x\sinh y$

Therefore, real part is $\cos x\cosh y$.

Correct option is (a)

Question 8. If C is the unit circle traversed clockwise, then the integral $\int_C \frac{dz}{z^2-3z}$ is **[DU 2017]**

(a) 0 (b) $\frac{\pi}{3}i$ (c) $\frac{2\pi}{3}i$ (d) $-\frac{2\pi}{3}i$

Soln: $f(z)=\frac{1}{z^2-3z}$ has simple poles are at z =0, 3 and only $z=0$ will lie within the unit circle.

Res. at $(z=0)=\lim_{z\to 0}\left[z\times\frac{1}{z(z-3)}\right]=\frac{1}{-3}$

Therefore, $I=-2\pi i\times-\frac{1}{3}=\frac{2\pi i}{3}$. (Since, the contour is in clockewise direction)

Correct option is (c)

Question 1. The sqaure roots of $(-15-8i)$ are **[H.C.U.-2010]**

(a) $1-4i,-1+4i$ (b) $1+4i,1-4i$ (c) $\sqrt{15}-\sqrt{8}i$ (d) $2\sqrt{2}-\sqrt{15}i$

Soln: Let $(-15-8i)^{1/2}=a+ib \Rightarrow -15-8i=(a+ib)^2=a^2-b^2+2iab$

$\therefore\ a^2-b^2=-15$ ... (i)

$2ab=-8$ ... (ii)

Solving equation (i) and (ii), we get $b=\pm 4,\ a=\mp 1$

$\therefore$ Square roots are $-1+4i$ and $1-4i$

Correct option is (a)

Question 2. $|z|=2$, where $z=x+iy$, represents a

(a) straight line (b) circle (c) ellipse (d) parabola **[H.C.U.-2011]**

Soln: $|z|=2 \Rightarrow |x+iy|=2\ ;\ \sqrt{x^2+y^2}=2 \Rightarrow x^2+y^2=4$

Hence, circle.

Correct option is (b)

Question 3. Given $\left(\frac{1-i}{1+i}\right)^{100}=a+ib$, the values of a and b are **[H.C.U.-2012]**

(a) $a=2,b=-1$ (b) $a=0,b=0$ (c) $a=0,b=1$ (d) $a=-1,b=2$

Soln: $\left(\frac{1-i}{1+i}\right)^{100}=a+ib \Rightarrow \left\{\left(\frac{1-i}{1+i}\right)\times\frac{1-i}{1-i}\right\}^{100}=\left(\frac{1-1-2i}{1+1}\right)^{100}=a+ib$

$\Rightarrow (-i)^{100}=a+ib \Rightarrow (i^2)^{50}=a+ib \Rightarrow 1=a+ib$

$\therefore\ a=1,\ b=0$

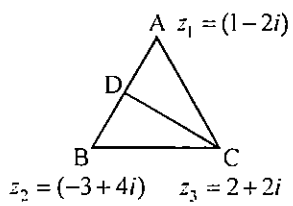
None of the options is correct.

Question 4. Let $(1-2i),(-3+4i),(2+2i)$ represent the three vertices A, B, C of a triangle ABC. the length of the meridian from C to the side AB is

(a) $\sqrt{10}$ (b) $\sqrt{12}$ (c) 6 (d) 8 **[H.C.U.-2013]**

Soln: From figure, $D=\frac{z_1+z_2}{2} \Rightarrow -\frac{2+2i}{2}=-1+i$

and, $CD=\sqrt{(2+1)^2+(2-1)^2}=\sqrt{10}$



Correct option is (a)

Question 5. Which of the following numbers does not lie on the unit circle $|z|=1$?

(a) $\left(\frac{1+i}{\sqrt{2}}\right)^{201}$ (b) $\left(\frac{1+i}{1-i}\right)^{57}$ (c) $\left(\frac{3+4i}{2-\sqrt{21}i}\right)$ (d) $\sqrt{\frac{1+2i}{3+2i}}$ **[H.C.U.-2013]**

Soln: Lets check option (c)

$$\frac{3+4i}{2-\sqrt{21}i}\times\frac{2-1\sqrt{21}i}{2+1\sqrt{21}i}=\frac{6+3\sqrt{21}i+8i-4\sqrt{21}}{4+21}=\frac{\left(6-4\sqrt{21}\right)+\left(8+3\sqrt{21}\right)i}{25}$$

Now, $\left|\frac{\left(6-4\sqrt{21}\right)+\left(8+3\sqrt{21}i\right)}{25}\right|>1$

Hence, this will not lie on the unit circle $|z|=1$.

Correct option is (c)

Question 6. The coefficient of x^4 in the expansion of $\sqrt{1+2x^2}$ is

(a) $-1/2$ (b) $-15/32$ (c) $-65/28$ (d) $-21/128$ **[H.C.U.-2013]**

Soln: $\left(1+2x^2\right)^{1/2}\Rightarrow 1+\frac{1}{2}2x^2+\frac{1}{2}\left(\frac{1}{2}-1\right)\times\frac{(2x^2)^2}{2!}+\cdots$

Therefore, co-efficient of x^4 is $\frac{1}{2}\times-\frac{1}{2}\times\frac{4x^4}{2}=-\frac{1}{2}$.

Correct option is (a)

Question 7. The real and imaginary parts of the function $f(z)=z\,e^z$ where $z\equiv x+iy$ respectively are

(a) $xe^x; ye^y$ (b) $xe^y; ye^x$ **[H.C.U.-2014]**

(c) $x\cos y; y\sin y$ (d) $[x\cos y-y\sin y]e^x; [y\cos y+x\sin y]e^x$

Soln: $f(z)=z\,e^z$

$=(x+iy)e^{x+iy}\Rightarrow(x+iy)e^x\,e^{iy}$

$=e^x(x+iy)(\cos y+i\sin y)$

$=e^x(x\cos y-y\sin y)+ie^x(y\cos y+x\sin y)$

Correct option is (d)

Question 8. The first three terms in the Taylor series expansion of $\sin x$ around $x=\frac{\pi}{2}$ are

(a) $\left(x-\frac{\pi}{2}\right)-\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}+\frac{\left(x-\frac{\pi}{2}\right)^5}{5!}$ (b) $\left(x-\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}+\frac{\left(x-\frac{\pi}{4}\right)^5}{5!}$

(c) $1-\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}$ (d) $1+\left(x-\frac{\pi}{2}\right)+\frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2$ **[H.C.U.-2014]**

Soln: $f(x)=\sin x \qquad f(\pi/2)=1$

$f'(x)=\cos x \qquad f'(\pi/2)=0$

$f''(x)=-\sin x \qquad f''(\pi/2)=-1$

$f'''(x)=-\cos x \qquad f'''(\pi/2)=0$ and so on.....

Therefore, Taylor expansion of $f(x)$ about $x=\frac{\pi}{2}$

$$=f\left(\frac{\pi}{2}\right)+\left(x-\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right)+\left(x-\frac{\pi}{2}\right)^2\frac{f''(\pi/2)}{2!}+\left(x-\frac{\pi}{2}\right)^3\frac{f'''(\pi/2)}{3!}+\cdots$$

$$=1-\left(\frac{x-\frac{\pi}{2}}{2!}\right)^2+\left(\frac{x-\frac{\pi}{4}}{4!}\right)^4-\cdots$$

Correct option is (c)

Question 9. In the complex z-plane, the equation $|z|=2|z-1|$ represents

(a) a circle of radius 2/3 (b) a circle of radius 1/2 **[H.C.U.-2015]**

(c) a circle of radius 3/2 (d) a straight line through (0, 0) and (1, 1)

Soln: $|z|=2|z-1| \Rightarrow |x+iy|=2|x+iy-1|$

$$\Rightarrow \sqrt{x^2+y^2}=2\sqrt{(x-1)^2+y^2} \Rightarrow x^2+y^2=4(x^2+1-2x+y^2)$$

$$\Rightarrow 3x^2+3y^2-8x+4=0 \Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\left(\frac{2}{3}\right)^2$$

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Correct option is (a)

Question 10. Given, $z=x+iy, e^{iz}$ reduces to

(a) $(\cos x+i\sin x)e^{-y}$ (b) $(\cos x+i\sin x)e^{y}$

(c) $(\cos x-i\sin x)(\cos y-i\sin y)$ (d) $(\cos x+i\sin x)(\cos y-i\sin y)$ **[H.C.U.-2016]**

Soln: $e^{iz}=e^{i(x+iy)}=e^{-y+ix}=e^{-y}e^{ix}=e^{-y}(\cos x+i\sin x)$

Correct option is (a)

Chapter 6

FOURIER TRANSFORM

6.1 Basic Formula

The fourier transform of $f(x)$ $[-\infty < x < \infty]$:

$$g(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Inverse fourier transform of $g(s)$ $[-\infty < s < \infty]$:

$$f(x) = F^{-1}\{g(s)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)\, e^{-isx}\, ds$$

Important Formula:

(i) $\int_0^{\infty} \frac{\sin ax}{x}\, dx = \frac{\pi}{2}$ (ii) $\int_0^{\infty} e^{-x^2}\, dx = \sqrt{\frac{\pi}{2}}$ (iii) $\int_0^{\infty} e^{-ax} \sin bx\, dx = \frac{b}{a^2 + b^2}$

(iv) $\int_0^{\infty} e^{-ax} \cos bx\, dx = \frac{a}{a^2 + b^2}$ (v) $\int_0^{\infty} e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ (vi) $\int_0^{\infty} e^{-ax^2} \cos bx\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{-b^2/4a}$

(vii) $\int_{-\infty}^{\infty} e^{-ax^2 + bx}\, dx = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$

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Example 1: Find the fourier transform of Gaussian distribution function $f(x) = Ne^{-ax^2}$ where N and 'a' are constant.

Soln: $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Ne^{-ax^2} e^{isx} dx = \frac{N}{\sqrt{2\pi}} \sqrt{\frac{\pi}{a}} e^{-s^2/4a} = \frac{N}{\sqrt{2a}} e^{-s^2/4a}$

Example 2: Find the fourier transform of the function $f(x) = \delta(x)$

Soln: $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \left[e^{isx}\right]_{x=0} = \frac{1}{\sqrt{2\pi}}$

Example 3: Find the fourier transform of $f(x)=e^{-a|x|} \quad (-\infty<x<\infty) \quad (a>0)$

Soln: $F\left[e^{-a|x|}\right]=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}e^{ax}e^{isx}dx+\int_{0}^{\infty}e^{-ax}e^{isx}dx\right]=\frac{1}{\sqrt{2\pi}}\left[\left(\frac{e^{(a+is)x}}{(a+is)}\right)_{-\infty}^{0}+\left(\frac{e^{-(a-is)x}}{(is-a)}\right)_{0}^{\infty}\right]$

$=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+is}+\frac{1}{a-is}\right]=\frac{2a}{\sqrt{2\pi}\left(s^2+a^2\right)}$

Example 4: If $F\left\{e^{-i2\pi vat}\right\}=f(\omega+a)$, then $F(\cos 2\pi vat)=?$

Soln: $F[\cos 2\pi vat]=F\frac{\left[e^{i2\pi vat}+e^{-i2\pi vat}\right]}{2}=\frac{1}{2}\left\{F\left[e^{i2\pi vat}\right]+F\left[e^{-i2\pi vat}\right]\right\}=\frac{1}{2}\left[f(\omega-a)+f(\omega+a)\right]$

Example 5: Find the fourier cosine transform of the function $f(x)=e^{-a^2x^2}$

Soln: $f_c(s)=F_c\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-a^2x^2}\cos sxdx=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}e^{-a^2x^2}\cos sxdx$

$=\frac{1}{2}\sqrt{\frac{2}{\pi}}$ Real part of $\int_{-\infty}^{\infty}e^{-a^2x^2}e^{isx}dx=\frac{1}{2}\sqrt{\frac{2}{\pi}}$ Real part of $\left(\sqrt{\frac{\pi}{a^2}}e^{\frac{i^2s^2}{4a^2}}\right)=\frac{1}{\sqrt{2a^2}}e^{-s^2/4a^2}$

6.2 Properties of Fourier Transform

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(1) Linearity theorem:

If $F(x)=a_1f_1(x)+a_2f_2(x)+....$ then, $g(s)=F\left[f(x)\right]=a_1g_1(s)+a_2g_2(s)+....$

(2) Change of scale property:

If $F\left[f(x)\right]=g(s)$, then $F\left\{f(ax)\right\}=\frac{1}{a}g(s/a)$

(3) Shifting property:

If $F\left[f(x)\right]=g(s)$, then $F\left\{f(x\pm a)\right\}=e^{\mp isa}g(s)$

(4) Modulation theorem:

If $F\left\{f(x)\right\}=g(s)$ then $F\left\{f(x)\cos ax\right\}=\frac{1}{2}g(s-a)+\frac{1}{2}g(s+a)$

(5) If $F\{f(x)\}=g(s)$, then $F\{f^*(x)\}=g^*(-s)$

(6) Derivative theorem:

If $F\{f(x)\}=g(s)$, then $F\left[\frac{df}{dx}\right]=-isg(s)$

If $F\{f(x)\}=g(s)$, then $F\left[\frac{d^nf}{dx^n}\right]=(-is)^n g(s)$

Example 6: Find Fourier transform of $f(x)=\begin{cases}x^2, & |x|<a\\ 0, & |x|>a\end{cases}$

Soln: $F\{f(x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}f(x)dx=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}e^{isx}x^2dx$

$$=\frac{1}{\sqrt{2\pi}}\left[\left(\frac{e^{isx}}{is}.x^2\right)_{x=-a}^{a}-\frac{2}{is}\int_{-a}^{a}xe^{isx}dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{a^2}{is}\left(e^{isa}-e^{-isa}\right)-\frac{2}{is}\left[\left(\frac{xe^{isx}}{is}\right)_{x=-a}^{a}-\frac{1}{is}\int_{-a}^{a}e^{isx}dx\right]\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{2a^2}{s}\sin(sa)+\frac{2a}{s^2}\cos(sa)-\frac{4}{s^3}\sin(sa)\right]$$

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Example 7: Find the complex fourier transform $e^{-|x|}$

Soln: $F\{e^{-|x|}\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-|x|}e^{isx}dx=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}e^{-|x|}e^{isx}dx+\int_{0}^{\infty}e^{-|x|}.e^{isx}dx\right]$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}e^{x}.e^{isx}dx+\int_{0}^{\infty}e^{-x(1-is)}dx\right]=\frac{1}{\sqrt{2\pi}}\left[\left[\frac{e^{x(1+is)}}{1+is}\right]_{x=-\infty}^{0}+\left[\frac{e^{-x(1-is)}}{-(1-is)}\right]_{x=0}^{\infty}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{1-is}+\frac{1}{1+is}\right]=\frac{1}{\sqrt{2\pi}}\left(\frac{2}{1+s^2}\right)$$

Example 8: Fourier transform of the derivative of the Dirac δ-function, namely $\delta'(x)$, is proportional to

(a) 0 (b) 1 (c) $\sin k$ (d) ik

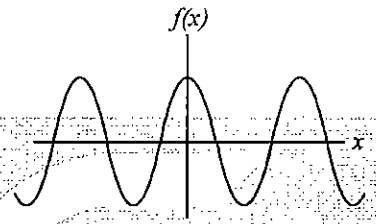
Soln: Fourier transform of $\delta'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta'(x) e^{ikx} dx$

Using the property of Dirac delta function $\int_{-\infty}^{\infty} f(x)\delta'(x-a)dx = -f'(a)$

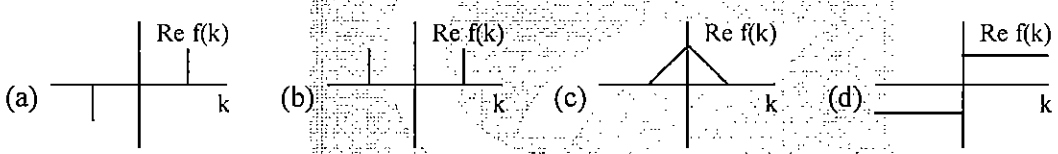
So, $F[\delta'(x)] = -\frac{1}{\sqrt{2\pi}}(ik) \quad \Rightarrow F[\delta'(x)] \propto ik$

Correct answer is (d)

Example 9: The graph of a real periodic function $f(x)$ for the range $[-\infty, \infty]$ is shown below



Which of the following graphs represents the real part of its Fourier transform?



Soln: The given graph in the question is of $f(x) = \cos \omega x$

$$F(\cos \omega x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cos \omega x \, e^{ikx} dx = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{+\infty} \frac{e^{i(k+\omega)x}}{2} dx + \int_{-\infty}^{+\infty} \frac{e^{i(k-\omega)x}}{2} dx \right)$$

$$\Rightarrow F(\cos \omega x) \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2}\delta(k+\omega) + \frac{1}{2}\delta(k-\omega) \right)$$

Therefore, the fourier transform of $f(x) = \cos x\omega x$ consist of two delta functions of same height at $k = \omega$ and $k = -\omega$.

Correct answer is (b)

Example 10: The Fourier transform of $f(x)$ is $\tilde{f}(k) = \int_{-\infty}^{\infty} dx \, e^{ikx} f(x)$. If

$f(x) = \alpha \delta(x) + \beta \dot{\delta}(x) + \gamma \ddot{\delta}(x)$, where $\delta(x)$ is the Dirac Delta function (and prime denotes derivative), what is $\tilde{f}(k)$?

(a) $\alpha + i\beta k + i\gamma k^2$ (b) $\alpha + \beta k - \gamma k^2$ (c) $\alpha - i\beta k - \gamma k^2$ (d) $i\alpha + \beta k - i\gamma k^2$

Soln: Given: Fourier transform of $f(x)$ is $\tilde{f}(x) = \int_{-\infty}^{\infty} dx \, e^{ikx} f(x)$

Fourier transform of $f''(x)$ is $-ik\,\tilde{f}(k)$

Fourier transform of $f''(x)$ is $(-ik)^2\,\tilde{f}(x) = -k^2\tilde{f}(x)$

So, $F[\delta(x)] = \int_{-\infty}^{\infty} dx\, e^{ikx}\delta(x) = 1$

$F[\delta'(x)] = -ik(1) = -ik$ and $F[\delta''(x)] = -k^2(1) = -k^2$

Therefore, $F[\alpha\delta(x) + \beta\delta'(x) + \gamma\delta''(x)] = \alpha - ik\beta - \gamma k^2$

Correct option is (c)

Example 11: The Heaviside function is defined as $H(t) = \begin{cases} +1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$ and its Fourier transform is given by $-2i/\omega$. The Fourier transform of $\frac{1}{2}\left[H\left(t+\frac{1}{2}\right) - H\left(t-\frac{1}{2}\right)\right]$ is

(a) $\dfrac{\sin\left(\frac{\omega}{2}\right)}{\omega/2}$ (b) $\dfrac{\cos\left(\frac{\omega}{2}\right)}{\omega/2}$ (c) $\sin\left(\frac{\omega}{2}\right)$ (d) 0

Soln: Shifting property:

If $F[f(t)] = f(\omega)$, then $F[f(t \pm a)] = e^{\mp i\omega a} f(\omega)$

$F[H(t)] = -\dfrac{2i}{\omega}$

$F\left\{\frac{1}{2}H\left(t+\frac{1}{2}\right) - \frac{1}{2}H\left(t-\frac{1}{2}\right)\right\}$

$= \frac{1}{2}e^{-i\omega/2}\left(-\frac{2i}{\omega}\right) - \frac{1}{2}e^{i\omega/2}\left(-\frac{2i}{\omega}\right)$

$= \frac{i}{\omega}\left[e^{i\omega/2} - e^{-i\omega/2}\right] = \frac{i}{\omega}\left(2i\sin\frac{\omega}{2}\right) = -\frac{\sin\frac{\omega}{2}}{(\omega/2)}$

Correct option is (a)

Chapter 7

LAPLACE TRANSFORM

7.1 Basic Formula

Laplace transform of function $f(x)$ defined in the interval $(0 < x < \infty)$:

$$f(s) = L[f(x)] = \int_0^\infty e^{-sx} f(x)\,dx$$

Important Laplace Transforms:

(a) $L(1) = \frac{1}{s}$

(b) $L(x^n) = \frac{n!}{s^{n+1}} (n = 0, 1, 2, ...)$

(c) $L(e^{ax}) = \frac{1}{s-a} (s > a)$

(d) $L(\cos ax) = \frac{s}{s^2 + a^2} (s > 0)$

(e) $L(\sin ax) = \frac{a}{s^2 + a^2} (s > 0)$

(f) $L(\cosh ax) = \frac{s}{s^2 - a^2} (s^2 > a^2)$

(g) $L(\sinh ax) = \frac{a}{s^2 - a^2} (s^2 > a^2)$

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7.2 Properties of Laplace Transform:

(1) Linear Property:

$$L[a_1 f_1(x) + a_2 f_2(x)] = a_1 L[f_1(x)] + a_2 L[f_2(x)]$$

(2) Shifting Property:

$$\text{If } L[f(x)] = g(s), \text{ then } L[e^{mx} f(x)] = g(s-m)$$

Example: $L[\cos ax] = \frac{s}{s^2 + a^2}$, then $L[e^{mx} \cos ax] = \frac{(s-m)}{(s-m)^2 + a^2}$

(3) Scaling Property:

$$\text{If } L[f(x)] = g(s), \text{ then } L[f(mx)] = \frac{1}{m} g\left(\frac{s}{m}\right)$$

Example: $L\left[\cos^2 x\right]=\dfrac{s^2+2}{s\left(s^2+4\right)}$,

then $L\left[\cos^2(mx)\right]=\dfrac{1}{m}\dfrac{(s/m)^2+2}{(s/m)\left[(s/m)^2+4\right]}=\dfrac{s^2+2m^2}{s\left(s^2+4m^2\right)}$

(4) If $L\left[f(x)\right]=g(s)$, then $L\left[x^n f(x)\right]=(-1)^n\dfrac{d^n}{ds^n}\left[g(s)\right]$

Example: $L[\sin 3x]=\dfrac{3}{s^2+9}$,

then $L[x\sin 3x]=(-1)^1\dfrac{d}{ds}\left[\dfrac{3}{s^2+9}\right]=(-1)\left[-\dfrac{3(2s)}{\left(s^2+9\right)^2}\right]=\dfrac{6s}{\left(s^2+9\right)^2}$

(5) If $L\left[f(x)\right]=g(s)$, then $L\left[\dfrac{f(x)}{x}\right]=\int_s^\infty g(s)\,ds$

Example: $L[\sin x]=\dfrac{1}{s^2+1}$.

then $L\left[\dfrac{\sin x}{x}\right]=\int_s^\infty\dfrac{1}{s^2+1}ds=\left[\tan^{-1}s\right]_s^\infty=\dfrac{\pi}{2}-\tan^{-1}s=\cot^{-1}s$

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(6) If $L\left[f(x)\right]=g(s)$, then $L\left[\int_0^t f(x)\,dx\right]=\dfrac{1}{s}g(s)$

Example: $L\left[\dfrac{\sin x}{x}\right]=\cot^{-1}s$

then $L\left[\int_0^t\dfrac{\sin x}{x}dx\right]=\dfrac{1}{s}L\left[\dfrac{\sin x}{x}\right]=\dfrac{1}{s}\cot^{-1}s$

(7) Laplace transforms of derivatives of a function:

If $L\left[f(x)\right]=g(s)$, then

(a) $L\left[f'(x)\right]=s.g(s)-f(x=0)$

(b) $L\left[f''(x)\right]=s^2.g(s)-f'(x=0)-s.f(x=0)$

(c) $L\left[f^{(n)}(x)\right]=s^n.g(s)-s^{n-1}.f(x=0)-s^{n-2}.f'(x=0).....s.f^{(n-2)}(x=0)-f^{(n-1)}.(x=0)$

(9) Laplace transform of periodic functions:

Suppose $f(x)$ is a periodic function having period 'T' such that

$$f(x+nT)=f(x) \quad \text{for } n=1,2,3.....$$

then $$L\left[f(x)\right]=\frac{1}{1-e^{-sT}}\int_0^T e^{-sx}f(x)dx$$

Example: Given $f(x)=x^2$ for $0<x<2$ and $f(x+2)=f(x)$

Then, $$L\left[f(x)\right]=\frac{1}{1-e^{-2s}}\int_0^2 e^{-sx}x^2dx$$

$$=\frac{1}{1-e^{-2s}}\left[-e^{-sx}\frac{x^2}{s}+2e^{-sx}\left(-\frac{x}{s^2}-\frac{1}{s^3}\right)\right]_0^2$$

$$=\frac{1}{1-e^{-2s}}\frac{2}{s^3}\left[1-e^{-2s}\left(1+2s+2s^2\right)\right]$$

(10) Laplace transform of unit step function:

$$u(x-a)=u_a(x)=\begin{cases}0 & \text{for } x<a\\ 1 & \text{for } x\geq a\end{cases}$$

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then $$L\left[u(x-a)\right]=\int_a^\infty e^{-sx}.1.dx=\frac{e^{-sa}}{s}$$

(11) Laplace transform of Dirac delta function:

$$L\left[\delta(x-a)\right]=\int_0^\infty e^{-sx}\delta(x-a)dx=e^{-sa}$$

Example 1: Find the laplace transform of $f(x)=(1+\cos 2x)$

Soln. $$L\left[f(x)\right]=\int_0^\infty e^{-sx}(1+\cos 2x)dx=\left(\frac{e^{-sx}}{-s}\right)_0^\infty+\int_0^\infty e^{-sx}\cos 2xdx$$

$$=\frac{1}{s}+\frac{s}{4+s^2}=\frac{2s^2+4}{s(s^2+4)}$$

Example 2: Find the laplace transform of $f(x)=2\sin 2x\cos 4x$

Soln. $L[f(x)]=L[2\sin 2x.\cos 4x]=L[\sin 6x-\sin 2x]=\left(\frac{6}{s^2+36}-\frac{2}{s^2+4}\right)$

Example 3: Find the laplace transform of $f(x)=e^{-x}(3\sinh 2x-5\cosh 2x)$

Soln. $L[f(x)]=L(3e^{-x}\sinh 2x)-L(5e^{-x}\cosh 2x)$

$$=3\frac{2}{(s+1)^2-4}-5.\frac{(s+1)}{(s+1)^2-4}=\frac{(1-5s)}{(s^2+2s-3)}$$ (Using 2nd property)

Example 4: Find the laplace transform of $f(x)=(1+xe^{-x})^3$

Soln. $f(x)=(1+xe^{-x})^3=1+x^3e^{-3x}+3xe^{-x}+3x^2e^{-2x}$

$L[f(x)]=L(1)+L(x^3e^{-3x})+L(3xe^{-x})+L(3x^2e^{-2x})$

$$=\frac{1}{s}+(-1)^3\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)+3(-1)^1\frac{d}{ds}\left(\frac{1}{s+1}\right)+3(-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right)$$ (Using 4th property)

$$=\frac{1}{s}+\frac{6}{(s+3)^4}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}$$

Example 5: Find the laplace transform of $f(x)=\frac{1-e^x}{x}$

Soln. $L\left(\frac{1-e^x}{x}\right)=\int_s^\infty\left(\frac{1}{s}-\frac{1}{s-1}\right)ds=\left[\ell n\,s-\ell n(s-1)\right]_s^\infty=\left[\ell n\frac{s}{s-1}\right]_s^\infty$ (Using 5th property)

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$$=\left[\ell n\,s-\ell n(s-1)\right]_s^\infty=\left[\ell n\frac{s}{s-1}\right]_s^\infty=\ell n\left(\frac{1}{1-\frac{1}{s}}\right)_s^\infty$$

$$=0-\ell n\frac{s}{s-1}=\ell n\frac{(s-1)}{s}$$

Example 6: Find the Laplace transform of the Bessel function of zeroth order $J_0(x)$.

Soln. $J_0(x)=\left[1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+.....\right]$

$$\Rightarrow L[J_0(x)]=\left[\frac{1}{s}-\frac{1}{2^2}\frac{2!}{s^3}+\frac{1}{2^2.4^2}\frac{4!}{s^5}-\frac{1}{2^2.4^2.6^2}\frac{6!}{s^7}+.....\right]$$

$$=\frac{1}{s}\left[1-\frac{1}{2}\left(\frac{1}{s^2}\right)+\frac{1.3}{2.4}\left(\frac{1}{s^2}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{s^2}\right)^3+.....\right]=\frac{1}{s}\left[1+\frac{1}{s^2}\right]^{-1/2}=\frac{1}{\sqrt{s^2+1}}$$

Example 7: Find the inverse laplace transform of

$$f(s)=\frac{s-2}{(s-2)^2+25}+\frac{s+4}{(s+4)^2+81}+\frac{1}{(s+2)^2+9}$$

Soln. $L^{-1}[f(s)]=L^{-1}\left[\frac{s-2}{(s-2)^2+25}\right]+L^{-1}\left[\frac{s+4}{(s+4)^2+81}\right]+L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$

$$=e^{2x}\cos 5x+e^{-4x}\cos 9x+\frac{e^{-2x}}{3}\sin 3x$$

Example 8: Find the inverse Laplace transform of $f(s)=\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right)$

Soln. $f(s)=\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right)=\frac{s+1}{s^2}-\frac{s+1}{s^2+1}=\frac{1}{s}+\frac{1}{s^2}-\frac{s}{s^2+1}-\frac{1}{s^2+1}$

$\Rightarrow L^{-1}\left[\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right)\right]=1+x-\cos x-\sin x$

Example 9: The inverse Laplace transform of $\frac{1}{s^2(s+1)}$ is

(a) $\frac{1}{2}t^2e^{-t}$ (b) $\frac{1}{2}t^2+1-e^{-t}$ (c) $t-1+e^{-t}$ (d) $\frac{1}{2}t^2(1-e^{-t})$

Soln. $f(s)=\frac{1}{s^2(s+1)}=\frac{(s+1)-s(s+1)+s^2}{s^2(s+1)}=\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}$

$\Rightarrow L^{-1}[f(s)]=t-1+e^{-t}$

Correct answer is (c).

Example 10: The Laplace transform of $6t^3+3\sin 4t$ is

(a) $\frac{36}{s^4}+\frac{12}{s^2+16}$ (b) $\frac{36}{s^4}+\frac{12}{s^2-16}$ (c) $\frac{18}{s^4}+\frac{12}{s^2-16}$ (d) $\frac{36}{s^3}+\frac{12}{s^2+16}$

Soln. $L[f(t)]=L[6t^3]+L[3\sin 4t]=6L[t^3]+3[L[\sin 4t]]$

Using $L[t^n]=\frac{n!}{s^{n+1}}$, $L[\sin at]=\frac{a}{s^2+a^2}$

We get, $L[f(t)]=6\cdot\frac{3!}{s^4}+3\cdot\frac{4}{s^2+16}=\frac{36}{s^4}+\frac{12}{s^2+16}$

Correct option is (a)

Example 11: The laplace transform of $f(t)=\sin \pi t$ is $F(s)=\frac{\pi}{\left(s^2+\pi^2\right)}, s>0$. Therefore, the laplace transform of $t \sin \pi t$ is:

(a) $\frac{\pi}{s^2\left(s^2+\pi^2\right)}$ (b) $\frac{2\pi}{s^2\left(s^2+\pi^2\right)}$ (c) $\frac{2\pi s}{\left(s^2+\pi^2\right)^2}$ (d) $\frac{2\pi}{\left(s^2+\pi^2\right)^2}$

Soln. **Property :** $L[f(x)]=f(s)$ then $L[x^n f(x)]=(-1)^n \frac{d^n}{ds^n}[f(s)]$

$$L[t \sin \pi t]=(-1)^1 \frac{d}{ds}[L(\sin \pi t)]=-\frac{d}{ds}\left[\frac{\pi}{s^2+\pi^2}\right]=\frac{2\pi s}{\left(s^2+\pi^2\right)^2}$$

Correct option is (c)

Example 12: If $f(x)=\begin{cases}0 & for\ x<3\\ x-3 & for\ x\geq 3\end{cases}$ then, the laplace transform of $f(x)$ is

(a) $s^{-2}e^{3s}$ (b) s^2e^{-3s} (c) s^{-2} (d) $s^{-2}e^{-3s}$

Soln. $L[f(x)]=\int_3^\infty e^{-sx}(x-3)dx=(x-3)\frac{e^{-sx}}{-s}\Big|_3^\infty-\int_3^\infty 1\cdot\frac{e^{-sx}}{-s}dx=\frac{e^{-sx}}{s^2}\Big|_3^\infty=\frac{e^{-3s}}{s^2}$

Correct option is (d)

Example 13: Which of the following pairs of the given function $F(t)$ and its Laplace transform $f(s)$ is not correct?

(a) $F(t)=\delta(t), f(s)=1$ (b) $F(t)=1, f(s)=\frac{1}{s}(s>0)$

(c) $F(t)=\sin kt, f(s)=\frac{s}{s^2+k^2}(s>0)$ (d) $F(t)=te^{kt}, f(s)=\frac{1}{(s-k)^2}(s>k, s>0)$

Soln. $L[\delta(t)]=\int_0^\infty \delta(t)e^{-sx}dx=1$

$$L[1]=\int_0^\infty 1\cdot e^{-sx}dx=\left[\frac{e^{-sx}}{-s}\right]_0^\infty=\frac{1}{s}$$

$$L[\sin kt]=\int_0^\infty \sin kt\, e^{-st}dt=\frac{k}{s^2+k^2}$$

$$L[te^{kt}]=L[t]_{s\to s-k}=\frac{1}{s^2}\Big|_{s\to s-k}=\frac{1}{(s-k)^2}$$

Correct option is (c)

Chapter 8

PROBABILITY

8.1 Basic Review of Probability

Definition:

Probability is the numerical measurement of degree of certainity of the occurrence of an event. If an event A can happen in m ways and fail in n ways then the probability of happening of event A will be equal to

$$P(A) = \frac{m}{m+n}$$

and probability of failing of event $A = P'(A) = \frac{n}{m+n}$

Therefore, $P(A) + P'(A) = 1$

Probability of any event A $= \frac{\text{number of events or ways favourable to A}}{\text{total number of events or ways}}$

(i) **Sample Space:** The set at all possible outcomes in a single performance of an random experiment is known as sample space. Each outcome is called sample point.

Example: Tossing a coin, we can get head or tail. Therefore, sample space S = (H, T)

(ii) **Random experiment :** These are experiments, in which result may be altogether different, even though they are performed under identical conditions.

Example: Tossing a coin, Throwing a die

(iii) **Trial and Event:** Performing a random experiment is called 'trial' and outcome is called 'event'.

Example: Tossing a coin is a trial and turning up head /tail are events.

(iv) **Equally like events:** Two events are said to be equally like events, if one of them cannot be expected in preference to the other.

Example: Drawing a card from a well shuffled pack,we may get any card, then all 52 different cases are equally like.

(v) **Independent Events:** If happening of one event does not influence occurrence of other, then they are independent events.

Example: If two coins are simultanously, then the event of getting head on first coin and the event of getting tail on second coin are independent.

(vi) Mutually Exclusive Events: Two events are known as mutually exclusive events, when the occurrence of one of them excludes the occurance of the other.

Example: Tossing a coin, we can get either head or tail but not both. Therefore, getting a head and getting a tail are mutually exclusive events.

(vii) Conditional Probability: The probability of happening of event A, such that B has already occurred, is called conditional probability of happening of A. It is usually denoted as P(A | B)

Example 1: What is the probability of throwing an even number in case of a dice?

Soln: $P = \dfrac{3\,[\text{getting 2, 4, 6 will be favourable cases}]}{6\,[\text{total no. of outcomes possibles are 1, 2, 3, 4, 5, 6}]}$

Example 2: A bag contains 7 red balls and 8 black balls. Find the probability of drawing a red ball.

Soln: $P = \dfrac{^{7}c_{1}[\text{No. of ways in which 1 red ball is chosen from 7 red balls}]}{^{15}c_{1}\,[\text{No. of ways in which 1 ball is chosen from 15 balls}]}$

Example 3: A unbiased coin is tossed two times. What is the probability of at least one head?

Soln: $P = \dfrac{3\,[\text{Outcomes having at least one head are: HT, TH, HH}]}{4\,[\text{Different outcomes possible: HH, HT, TH, TT.}]}$

Theorem 1: Probability that either event A or event B or both will occur (when they are mutually exclusive) is given by

$$P(A \cup B) = P(A) + P(B)$$

Theorem 2: Probability that either event A or event B or both will occur (when they are not mutually exclusive) is given by,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Theorem 3: Probability that both A, B events occur (when they are independent events) is given by,

$$P(A \cap B) = P(A)\,P(B)$$

Theorem 4: Probability that both A, B events occur (when they ar not independent events) is given by,

$$P(A \cap B) = P(B \mid A).P(A) \quad [\text{Here A occurs first, then } B]$$
$$= P(A \mid B).P(B) \quad [\text{Here B occurs first, then } A]$$

Theorem 5: For every event A, $0 \le P(A) \le 1$ i.e. a probability is between 0 and 1.

Theorem 6: $P(\varnothing) = 0$ i.e. the impossible event has probability zero.

Theorem 7: If A' is the complement of A, then $P(A') = 1 - P(A)$

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Example 4: Find the probability of throwing 9 with two dice.

Soln: Total number of possible ways of throwing two dice $= 6 \times 6 = 36$
Number of ways getting 9 i.e. (3+6), (4+5), (5+4), (6+3) = 4
Therefore, the required probability = 4/36 =1/9

Example 5: What is the probability that a leap year, selected at random, will contain 53 sundays?

Soln: Every year has minimum of 52 weeks = 364 days i.e. there will be 52 sundays, 52 mondays, ... etc.
Since a leap year contains 366 days, there are 2 extra days. If the year starts on either saturday or sunday, we will get the 53rd sunday. Therefore, probability of getting 53 sundays in a leap year is $\frac{2}{7}$.

Example 6: A unbiased coin is tossed 3 times. What is the probability of obtaining two heads?

Soln: The possible outcomes are HHH, HHT, HTH, THH, HTT, THT, TTH, TTT.

The favorable outcomes are HHT, HTH, THH.

The required probability of getting 2 heads is $\frac{3}{8}$.

Example 7: A problem is given to three students A, B and C whose chances of solving it are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that this problem will be solved?

Soln: Since $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}, P(C)=\frac{1}{4}$, then $P'(A)=\frac{1}{2}, P'(B)=\frac{2}{3}, P'(C)=\frac{3}{4}$

The problem will not be solved if all three of them will not be able to solve the problem.

Then, probability that the problem will not be solved $= P'(A)\,P'(B)\,P'(C)=\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}=\frac{1}{4}$

Therefore, the probability that the problem will be solved $= 1-\frac{1}{4}=\frac{3}{4}$

Example 8: From a pack of 52 cards one are drawn at random. What is the probability that the card is either a king or a spade?

Soln: Probability that one spade will be choosen from a pack of 52 cards $=\frac{13}{52}=\frac{1}{4}$

Probability that one king will be choosen from a pack of 52 cards $=\frac{4}{52}=\frac{1}{13}$

Since, there is one card which is both king and spade. Then, the probability that the choosen card is either a king or a spade $=\frac{1}{4}+\frac{1}{13}-\frac{1}{52}=\frac{4}{13}$

Example 9: Let $P\begin{bmatrix} 1 & a \\ b & c \end{bmatrix}$, where a, b, c are chosen randomly from the set {1, 2, 3, 4, 5}. What is the probability that P is singular?

Soln: $P=\begin{bmatrix} 1 & a \\ b & c \end{bmatrix}$; for the matrix to be singular $|P| = c - ab = 0$

$$\text{Required probability} = \frac{\text{number of singular matrices}}{\text{total number of matrices formed}}$$

Each a, b, c can be chosen in five ways, therefore, total number of matrices formed = 5×5×5=125

For the following combinations of a, b, c , $|P| = c - ab = 0$:

c	1	2 2	3 3	4 4 4	5 5
a	1	1 2	3 1	1 4 2	1 5
b	1	2 1	1 3	4 1 2	5 1

So, (Probability) $P = \frac{10}{125} = \frac{2}{25}$

Example 10: A six faced die is a biased one. it is thrice more likely to show an odd number than to show an even number. It is thrown twice. What is the probability that the sum of the numbers in the two throws is even?

Soln: $\text{Prob.}(\text{odd}) = \text{Prob.}(\text{even})$

$\text{Prob.}(\text{odd}) + \text{Prob.}(\text{even}) = 1$

$\Rightarrow 3\text{Prob.}(\text{even}) + \text{Prob.}(\text{even}) = 1 \Rightarrow \text{Prob.}(\text{even}) = \frac{1}{4}$ and $\text{Prob.}(\text{odd}) = \frac{3}{4}$

Sum of the numbers will be even if both numbers will be either even or odd.

$$\text{Required probability} = \text{Prob.}(\text{even-even}) + \text{Prob.}(\text{odd-odd}) = \left(\frac{1}{4}\times\frac{1}{4}\right) + \left(\frac{3}{4}\times\frac{3}{4}\right) = \frac{5}{8}$$

8.2 Baye's Theorem

If B_1, B_2, B_3...B_n are mutually exclusive events (with non-zero probability) of an random experiment, then for any arbitrary event A of the sample space of above random experiment, the probability of event B_k such that A has already occured is

$$P(B_k \mid A) = \frac{P(B_k)\,P(A \mid B_k)}{\sum_{i=1}^{n} P(B_i)P(A \mid B_i)}$$

Example 11: Suppose bag 1 contains 2 white, 3 red balls and bag 2 contains 3 white and 4 red balls. If a bag is selected at random and a ball is drawn from it. If the drawn ball is white, what is probability that the ball is from bag 1?

Soln: Event A: The drawn ball is white, Event B_1: Choosing bag 1, Event B_2: Choosing bag 2

Probability that the ball is chosen from bag 1 given that the ball is white =

$$P(B_1 \mid A) = \frac{P(B_1)\,P(A \mid B_1)}{\sum_{i=1}^{2} P(B_i)P(A \mid B_i)} = \frac{P(B_1)\,P(A \mid B_1)}{P(B_1)\,P(A \mid B_1) + P(B_2)\,P(A \mid B_2)} = \frac{\frac{1}{2} \times \frac{{}^{2}c_1}{{}^{5}c_1}}{\frac{1}{2} \times \frac{{}^{2}c_1}{{}^{5}c_1} + \frac{1}{2} \times \frac{{}^{3}c_1}{{}^{7}c_1}} = \frac{14}{29}$$

8.3 Random Variable

It is defined as a real number 'x' connected with the outcomes of a random experiment. There are of two types as follows:

(i) Discrete random variables

(ii) Continuous random variables

Discrete Random Variable:

In this case the random variable x will take discrete values.

Example:

Two coins are tossed together. The sample place will be

S = {HH, HT, TH, TT}

Let X be a ranodm variable which denote the number of heads that we have obtained. Then, X can take values 0, 1, 2

Continuous Random Variable:

In this case random variable x can take contiuous values.

Example : Height and weight of a students in a class is a continuous random variable

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8.4 Probability Distribution

This gives the all possible outcomes of a random experiment with their respective probabilities. These are of two types

(i) Discrete probability distribution

(ii) Continuous probability distribution

Discrete probability distribution:

Let X is a discrete random variable which can take the following discrete values $X_i\,(i = 1,2,3,.....n)$ with corresponding probabilities $P(X_i)(i = 1,2,3.....n)$ such that

$$\sum_{i=1}^{n} P(x_i) = 1$$

then it will be discrete probability distribution with $P(x_i)$ as a probability distribution function.

(i) Mean of the distribution:

It can be defined as the average or expected value of the random variable 'x' and given by

$$E(x)=\mu=\langle x\rangle=\sum_{i=1}^{n}x_iP(x_i)$$

(ii) Variance and standard deviation of the distribution:

It can be defined as the measurement of deviation of the random variable from its mean value and given by

$$\text{variance } \sigma^2(x)=\sum_{i=1}^{n}(x_i-\mu)^2P(x_i)=\langle x^2\rangle-\langle x\rangle^2$$

$$\text{Standard Deviation, } \sigma(x)=\sqrt{\langle x^2\rangle-\langle x\rangle^2}$$

Note:

(i) $E(K\,x)=K\,E(x)$

(ii) $E(x+y)=E(x)+E(y)$

(iii) $E(xy)=E(x)\cdot E(y)$

(iv) $\sigma^2(x+y)=\sigma^2(x)+\sigma^2(y)$

Example 12: A dice thrown and the outcomes are observed. Find the mean, variance, standard deviation of the random experiment.

Soln: The probability distirbution of the random experiment is as follows

x_i	1	2	3	4	5	6
$P(x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\text{Mean } \mu=\sum_{i=1}^{6}x_iP(x_i)=1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=3.5$$

Variance

$$\sigma^2=\sum_{i=1}^{6}(x_i-\mu)^2P(x_i)$$

$$=(1-3.5)^2\times\frac{1}{6}+(2-3.5)^2\times\frac{1}{6}+(3-3.5)^2\times\frac{1}{6}+(4-3.5)^2\times\frac{1}{6}+(5-3.5)^2\times\frac{1}{6}+(6-3.5)^2\times\frac{1}{6}=2.9$$

Standard deviation $\sigma=\sqrt{2.9}\simeq1.7$

Continuous probability distribution:

Suppose x is a continuous random variable and the continuous probability distribution of x is defined by a probability density function or probability distribution function $f(x)$ such that

$$\int_{-\infty}^{\infty}f(x)\,dx=1$$

i.e. probability of finding the continuous random variable x from $-\infty$ to $+\infty$ is equal to 1.

Mean of the distribution (μ) $= \langle x \rangle = \int_{-\infty}^{\infty} xf(x)\,dx$

Variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

Standard deviation $\sigma = \left[\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx\right]^{1/2}$

Example 13: Suppose, 'x' is a continuos random varible with probability density function given by,

$$f(x) = kx\,;(0 \le x < 2)$$
$$= 2k\,;(2 \le x < 4)$$
$$= -kx + 6k\,;(4 \le x < 6)$$

What is the value of k and mean of the distribution?

Soln: $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_0^2 kx\,dx + \int_2^4 2k\,dx + \int_4^6 (-kx+6k)\,dx = 1 \qquad \Rightarrow k = \frac{1}{8}$$

Mean $\mu = \int_{-\infty}^{\infty} xf(x).dx = \int_0^2 k.x^2\,dx + \int_2^4 2kx\,dx + \int_4^6 (-kx+6x)\,x\,dx = 3$

8.5 Binomial Distribution

Binomial distribution can be used under the following conditions

(a) A random experiment is repeated many times but the number of trials will be finite

(b) There are only two possible outcomes i.e. success and failure, for each trial.

(c) All trials are independent of each other

If a random experiment is repeated 'n' times (i.e. there are 'n' trials of a random experiment), the probability that event A will occur 'r' times is

$$P(r) = {}^nC_r\ p^r q^{n-r}$$

where, n = total number of independent trials
r = number of success
p = probability of success in a single trial
q = probability of failure in a single trial.

Mean of distribution $= \mu = \langle x \rangle = np$

Standard Deviation of the ditribution $\sigma = \sqrt{npq}$

Variance of the distribution $\sigma^2 = npq$

Note : (i) $\langle x \rangle = np$

(ii) $\langle x^2 \rangle = np(1-p+np)$

(iii) $\langle x^3 \rangle = np(1-3p+3np+2p^2-3np^2+n^2p^2)$

Example 14: Find the proability of getting at least 4 head in 5 tosses of a fair coin.

Soln: Number of trials $n = 5$, number of success $r = 4$ or 5.

$$\text{Required Probability} = P(4)+P(5) = {}^5C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{5-4} + {}^5C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{5-5} = 5.\left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5 = \frac{3}{16}$$

Example 15: Assuming that each face of six faced dice is equally likely to land uppermost. Consider a game which involves the tossing of five such dices. Calculate the probability that the number 4 appears uppermost.

(i) In exactly one dice. (ii) In exactly two dice.

(iii) In exactly five dice (iv) in atleast one dice.

Soln: (i) $P = {}^5c_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4 = 0.4$ (ii) $P = {}^5c_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = 0.16$ (iii) $P = {}^5c_5\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^0 = \left(\frac{1}{6}\right)^5$

(iv) At least one means it will 1,2,3,4,5 Therefore, $P = 1 - {}^5c_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5 = 0.6$

8.6 Poisson Distribution

Poisson distribution is a limiting form of binomial distribution. It can be used under the following conditions:

(a) the number of trials are infinitely large i.e $n \to \infty$

(b) the constant probability of success for each trial is infinitely small i.e. $p \to 0$

(c) np is a finite quantity

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If a random experiment is repeated 'n' times, then the probability that event A will occur 'r' times

$$P(r) = \frac{e^{-m} \cdot m^r}{r!}$$

where 'm' is the mean of the distribution.

Mean of the distribution, $\mu = \langle x \rangle = np$

Variance of the ditribution $\sigma^2 = np$

Standard Deviation of the distribution $\sigma = \sqrt{np}$

Note: (i) $\langle x \rangle = np$

(ii) $\langle x^2 \rangle = np(1+np)$

(iii) $\langle x^3 \rangle = np(1+3np+n^2p^2)$

Example 16: Ten percent of the tools produced in a certain manufacturing process turn out to be defective. Find the probability that in a sample of 10 tools chosen at random, exactly 2 will be defective, by using (a) the binomial distribution, (b) the poisson distribution.

Soln: (a) The probability of a defective tool is p = 0.1. Let X denote the number of defective tools out of 10 chosen. Then, according to the binomial distribution,

$$P(r=2)=\left({}^{10}c_2\right)(0.1)^2(0.9)^8=0.1937 \text{ or } 0.19$$

(b) We have $m = np = (10)(0.1) = 1$. Then, according to the Poisson distribution,

$$P(r=2)=\frac{m^r e^{-m}}{r!}=\frac{(1)^2 e^{-1}}{2!}=0.1839 \text{ or } 0.18$$

Example 17: A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a poisson's distribution with mean 1.5. Calculate the number of days in a year on which (i) car is not used (ii) The number of days in a year on which some demand is refused.

Soln: Mean of the distribution 'm' = 1.5

(i) If the car is not used, then demand r=0

$$P(r)=\frac{e^{-m}m^r}{r!} \Rightarrow P(0)=\frac{e^{-1.5}(1.5)^0}{0!}=e^{-1.5}=0.2231$$

Therefore, number of days in a year when there is no demand of car. $= 365 \times 0.2231 = 81.4315 = 81$ days

(ii) Some demand is refused if the number of demands is more than 2 i.e. r > 2.

$$P(r>2)=p(3)+p(4)+\ldots=1-\left[P(0)+P(1)+P(2)\right]$$

$$=1-\left[\frac{e^{-1.5}(1.5)^0}{0!}+\frac{e^{-1.5}(1.5)^1}{1!}+\frac{e^{-1.5}(1.5)^2}{2!}\right]=0.1912625$$

Therefore, number of days in a year when some demand is refused $= 365 \times 0.1912625 = 69.81 = 70$ days

Example 18: If the probability of unfavorable reaction from a certain injection is 0.001, what is the probability that out of 2000 individuals, more than 2 will suffer from unfavorable reaction?

Soln: Number of trials n = 2000

Probability of success in single trial p = 0.001

Probability that more than two persons will suffer from the unfavourable reaction =

$$P(r>2)=1-\left[P(0)+P(1)+P(2)\right]=1-\left[\frac{e^{-2}2^0}{0!}+\frac{e^{-2}2^1}{1!}+\frac{e^{-2}2^2}{2!}\right]=1-\frac{5}{e^2}$$

8.7 Gaussian Distribution

The Gaussian probability distribution function for a continuous random variable x is given by

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

where, μ = mean of distribution $(-\infty < \mu < \infty)$

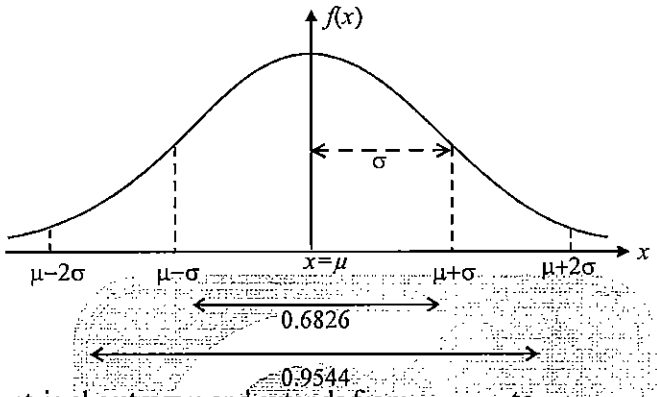
σ = S.D. of the distribution $(\sigma > 0)$

Replacing $\frac{x-\mu}{\sigma}$ by z, we get the probability distribution function for the normal distribution.

i.e. $f(z)=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]$

where, mean of the distribution $\mu=0$ and standard deviation $\sigma=1$

Normal curve and its properties:



(a) The curve is symmetric about $x=\mu$ and extends from $x=-\infty$ to $x=\infty$

(b) The curve attains a peak about $x=\mu$ and the peak value $f(x)|_{max}=\frac{1}{\sqrt{2\pi}\sigma}$

(c) As we change x on both sides from μ, y decreases rapidly.

(d) Area under the curve is the measurement of probability for the ranodm variable 'x'.

$$P(\mu-\sigma\le x\le\mu+\sigma)=0.6826$$
$$P(\mu-2\sigma\le x\le\mu+2\sigma)=0.9544$$
$$P(\mu-2\sigma\le x\le\mu+3\sigma)=0.9974$$

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Note:

(i) $\langle x\rangle=\mu$

(ii) $\langle x^2\rangle=\mu^2+\sigma^2$

(iii) $\langle x^3\rangle=(\mu^2+3\sigma^2)\mu$

(iv) $\langle x^4\rangle=\mu^4+6\mu^2\sigma^2+3\sigma^4$

Example 19: A manufacturer produces airmail envelopes whose weight is normally distributed with a mean 1.95 gm and standard deviation 0.05 gm. The envelopes are sold in lots of 1000. How many envelopes in a lot will heavier than 2 gm?

Soln: Maximum number of envelops have weight $\mu=1.95$ gram.

Since the standard deviation in the weight of envelops $=\sigma=0.05$g $\Rightarrow \mu+\sigma=2$ g, $\mu-\sigma=1.90$ g

Percentage of envelops having weight between 1.90 gram and 2 gram = 68.7 %

Percentage of envelops having weight less than 1.90 g and more 2 gram = 100% – 68.7 % = 31.3%

Percentage of envelops having weight more than 2 grams = 15.65 % (since the normal curve is symmetric.)

Number of envelops having weight more than 2 grams = $1000 \times 15.65 \approx 156$.

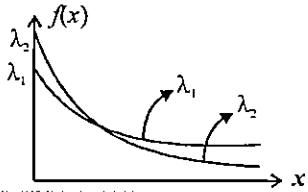
8.8 Exponential Distribution

The probability distribution function is given as follows

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0$$
$$= 0 \quad \text{for } x < 0$$

where λ is known as rate parameter (+ve) and its denotes the decay rate of probability.

If λ_1, λ_2 are two rate parameter and $\lambda_2 > \lambda_1$, then



Mean : $\mu = \langle x \rangle = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$\langle x \rangle = \int_{-\infty}^{\infty} x.\lambda e^{-\lambda x} dx = \frac{1}{\lambda}\int_{-\infty}^{\infty} t\, e^{-t} dt = \frac{1}{\lambda}$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \lambda e^{-\lambda x} dx = \frac{1}{\lambda^2}$$

TIFR PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. A 100 page book is known to have 200 printing errors distributed randomly through the pages. The probability that one of the pages will be found to be completely free of errors is closest to **[TIFR 2011]**

(a) 67% (b) 50% (c) 25% (d) 13%

Soln: Since, 100 page contains 200 error hence, on an average, error per page = 2.

Here, R.E. = checking 1 page for error.

Success = getting error in a page.

From poission distribution, Probability of not getting an error in a page ($r = 0$)

$$P(r) = \frac{e^{-nP}(nP)^r}{r!}$$

Here, r = number of times of success, nP = average error per page.

Therefore, $r = 0$, $nP = 2$

Therefore, $P(r=0) = \frac{e^{-2} \times (2)^0}{0!} = \frac{1}{e^2} \simeq 13\,\%$.

Correct option is (d)

Question 2. The probability function for a variable x which assumes only positive values is

$$f(x) = x \exp\left(-\frac{x}{\lambda}\right)$$

where $\lambda > 0$. The ratio $\langle x \rangle / \hat{x}$, where $\hat{x}$ is the most probable value and $\langle x \rangle$ is the mean value of the variable x, is **[TIFR 2014]**

(a) 2 (b) $\frac{1+\lambda}{1-\lambda}$ (c) $\frac{1}{\lambda}$ (d) 1

Soln: $f(x) = x\exp\left(\frac{-x}{\lambda}\right) \rightarrow$ Probability function

$$\text{Mean value } \langle x\rangle = \frac{\int_0^\infty xf(x)dx}{\int_0^\infty f(x)dx} = \frac{\int_0^\infty x^2\cdot e^{-x/\lambda}dx}{\int_0^\infty xe^{-x/\lambda}dx} = \frac{\int_0^\infty \lambda^2p^2\cdot e^{-p}\cdot\lambda dp}{\int_0^\infty e^{-p}\cdot\lambda^2 pdp} = \frac{\lambda\int_0^\infty e^{-p}\cdot p^2dp}{\int_0^\infty e^{-p}\cdot pdp}$$

Assuming, $\frac{x}{\lambda} = p, \Rightarrow x = \lambda p \Rightarrow dx = \lambda dp$

[Using gamma integral, $\int_0^\infty e^{-p}\cdot p^{n-1}dp = \Gamma n$] $= \frac{\lambda\Gamma 3}{\Gamma 2} = 2\lambda$

At $x = \hat{x}$ (most probable value), $\frac{df}{dx} = 0$

$$\Rightarrow \quad e^{-x/\lambda} + x\cdot e^{-x/\lambda}\cdot\left(-\frac{1}{\lambda}\right) = 0 \quad \Rightarrow \quad e^{-x/\lambda}\left(1-\frac{x}{\lambda}\right) = 0 \Rightarrow x = \hat{x} = \lambda$$

Therefore, $\frac{\langle x\rangle}{\hat{x}} = 2$

Correct option is (a)

Question 3. A random number generator outputs +1 or -1 with equal probability every time it is run. After it is run 6 times, what is the probability that the sum of the answers generated is zero? Assume that the individual runs are independent of each other. **[TIFR 2015]**

(a) 1/2 (b) 5/6 (c) 5/16 (d) 15/32

Soln: R.E. = running a random number generator, Number of trials $n = 6$

p = getting '+1' in single run and q = getting '–1' in single run.

Hence, $p = \frac{1}{2}$, $q = \frac{1}{2}$

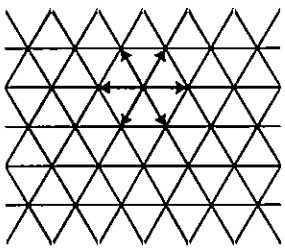
Sum will be zero, if (+1) occurs 3 times and (–1) occurs 3 times.

Hence, required probability $P(r) = {}^nC_3\, p^r\, q^{n-r}$

Here, $P(r=3) = {}^6C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3 = \frac{5}{16}$.

Correct option is (c)

Question 4. In a triangular lattice a particle moves from a lattice point to any of its 6 neighbouring points with equal probability, as shown in the figure on the right. **[TIFR 2016]**



The probability that the particle is back at its starting point after 3 moves is

(a) 5/18 (b) 1/6 (c) 1/18 (d) 1/36

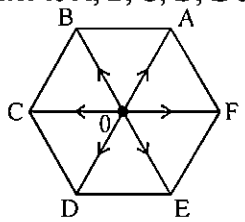
Soln: Total number of ways along which particle can move in 3 steps is $6\times6\times6=216$.

To return to the starting point after 3 steps,

Number of ways to take 1st step 6. (Particle may go either to A, B, C, D, E or F).

Number of ways to take 2nd step is 2.

Number of ways to take 3rd step 1.



Therefore, required probability $=\dfrac{6\times2\times1}{216}=\dfrac{1}{18}$

Correct option is (c).

JEST PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. An unbiased die is cast twice. The probability that the positive difference (bigger-smaller) between the two numbers is 2 is **[JEST 2012]**

(a) 1/9 (b) 2/9 (c) 1/6 (d) 1/3

Soln: Sample space $=S=36$.

$E=$ event of getting the difference 2.

$\{(3,1),(4,2),(5,3),(6,4),(1,3),(2,4),(3,5),(4,6)\}$

$n(E)=8$

Therefore, required probability $P=\dfrac{8}{36}=\dfrac{2}{9}$.

Correct option is (b)

Question 2. A box contains 100 coins out of which 99 fair coins and 1 is a double-headed coin. Suppose you choose a coin at random and toss it 3 times. It turns out that the results of all 3 tosses are heads. What is the probability that the coin you have drawn is the double-headed one?

[JEST 2013]

(a) 0.99 (b) 0.925 (c) 0.075 (d) 0.01

Soln: Total 100 coins = 99 fair coins + 1 double headed coin

Suppose Event A : getting 3 heads in 3 tosses

Event B_1 : selecting a fair coin

Event B_2 : selecting a double headed coin.

$$P(B_2\mid A)=\frac{P(B_2)P(A\mid B_2)}{P(B_1)P(A\mid B_1)+P(B_2)P(A\mid B_2)}=\frac{\frac{{}^1C_1}{{}^{100}C_1}\times1}{\frac{{}^{99}C_1}{{}^{100}C_1}\times\frac{1}{8}+\frac{{}^1C_1}{{}^{100}C_1}\times1}=0.075$$

Correct option is (c)

Question 3. There are on average 20 buses per hour at a point, but at random times. The probability that there are no buses in five minutes is closest to **[JEST 2013]**

(a) 0.07 (b) 0.60 (c) 0.36 (d) 0.19

Soln: On an average number of buses in one $\min = \frac{20}{60} = \frac{1}{3}$.

Hence, number of buses (in average) in 5 min = 5/3.
R.E. = checking for bus for five minutes.
Success = getting a bus.

Therefore, required probability $P(r) = \frac{e^{-nP}(nP)^r}{r!}$.

Here r = number of times of success = 0.
nP = average number of buses in 5 min = 5/3.

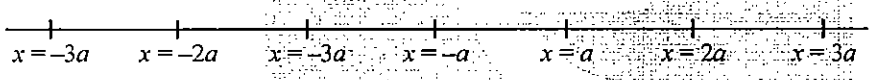
Therefore, $P(r=0) = \frac{e^{-5/3}(5/3)^0}{0!} = e^{-5/3} \simeq 0.18$.

Correct option is (d)

Question 4. Two drunks start out together at the origin, each having equal probability of making a step simultaneously to the left or right along the x-axis. The probability that they meet after n steps is **[JEST 2014]**

(a) $\frac{1}{4^n}\frac{2n!}{n!^2}$ (b) $\frac{1}{2^n}\frac{2n!}{n!^2}$ (c) $\frac{1}{2^n}2n!$ (d) $\frac{1}{4^n}n!$

Soln:



Let us keep random walker 1 at rest and let random walker 2 takes '$2n$' steps. Then random walker 2 will return back to origin or at random walker 1, if he takes n steps right and 'n' steps left.
Total steps = $2n$
p = probability of taking a step right = 1/2.
q = probability of taking a step left = 1/2.

Therefore, required probability $p(r=n) = {}^{2n}C_n\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{2n-n} = \frac{1}{4^n}\frac{(2n)!}{(n!)^2}$.

Correct option is (a)

Question 5. If the distribution function of x is $f(x) = xe^{-x/\lambda}$ over the interval $0 < x < \infty$, then mean value of x is

[JEST 2014]

(a) λ (b) 2λ (c) $\lambda/2$ (d) 0

Soln: $f(x) = xe^{-x/\lambda}$

$\langle x \rangle$ = mean value of $x = \frac{\int_{-\infty}^{\infty} x f(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx}$

Here, $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} x e^{-x/\lambda} dx = \left(x\left(-\lambda e^{-x/\lambda}\right) - \lambda^2\left(e^{-x/\lambda}\right)\right)_0^{\infty} = \lambda^2$

and $\int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x^2 e^{-x/\lambda}\,dx = \left(x^2\left(-\lambda e^{-x/\lambda}\right)\right)_0^{\infty} + 2\lambda\int_0^{\infty} x e^{-x/\lambda}\,dx$

$\Rightarrow \quad 0 + 2\lambda \times \lambda^2 = 2\lambda^3.$

Therefore, $\langle x\rangle = \dfrac{2\lambda^3}{\lambda^2} = 2\lambda.$

Question 6. If two ideal dice are rolled once, what is the probability of getting atleast one '6'?

[JEST 2015]

(a) 11/36 (b) 1/36 (c) 10/36 (d) 5/36

Soln: $n(s) = 36$

E = event of getting at least one six.

$= \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,1), (6,2), (6,3), (6,4), (6,5)\}$

$n(E) = 11$

Therefore, required probability = 11/36.

Correct option is (a)

Question 7. You receive on average 5 emairs per day during a 365-days year. The number of days on average on which you do not receive any emails in that year are **[JEST 2016]**

(a) More than 5 (b) More than 2 (c) 1 (d) None of the above

Soln: Average emails per day = 5

R.E. = Checking that email is received or not

Success = Getting an email in a day

Therefore, $P(r) = \dfrac{e^{-nP}(nP)^r}{r!}$

Here, $r = 0$, $nP = 5$

$P(r=0) = \dfrac{e^{-5}(5)^0}{0!} = e^{-5}.$

Therefore, number of days on average on which we do not receive any emails.

$= e^{-5} \times 365 =$ more than 2 days.

Correct option is (b)

Question 8. The mean value of random variable x with probability density $p(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(x^2 + \mu x\right)/2\sigma^2\right]$, is

[JEST 2016]

(a) 0 (b) $\mu/2$ (c) $-\mu/2$ (d) σ

Soln: We have probability function, $P(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left(x^2+\mu x\right)}{2\sigma^2}}$

Now, $\frac{dP}{dx}=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left(x^2+\mu x\right)}{2\sigma^2}}\left(-2x-\mu\right)$

For $x=x_0$, $\left.\frac{dP}{dx}\right|_{x=x_0}=0\Rightarrow\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left(x_0^2+\mu x_0\right)}{2\sigma^2}(-2x_0-\mu)}=0 \quad \Rightarrow x_0=-\frac{\mu}{2}$

For gaussian distribution most probable and mean value are equal

Therefore, mean value of x is $-\frac{\mu}{2}$

Correct option is (c)

Question 9. Suppose that we toss two fair coins hundred times each. The probability that the same number of *heads* occur for both coins at the end of the experiment is **[JEST 2017]**

(a) $\left(\frac{1}{4}\right)^{100}\sum_{n=0}^{100}\binom{100}{n}$ (b) $2\left(\frac{1}{4}\right)^{100}\sum_{n=0}^{100}\binom{100}{n}^2$

(c) $\frac{1}{2}\left(\frac{1}{4}\right)^{100}\sum_{n=0}^{100}\binom{100}{n}^2$ (d) $\left(\frac{1}{4}\right)^{100}\sum_{n=0}^{100}\binom{100}{n}^2$

Soln: Required probability

$$=\sum_{n_1}{}^{100}C_{n_1}\left(\frac{1}{2}\right)^{n_1}\left(\frac{1}{2}\right)^{100-n_1}\sum_{n_2}{}^{100}C_{n_2}\left(\frac{1}{2}\right)^{n_2}\left(\frac{1}{2}\right)^{100-n_2}$$

$$=\left(\frac{1}{4}\right)^{100}\sum_{n_1}{}^{100}C_{n_1}\sum_{n_2}{}^{100}C_{n_2}$$

For $n_1=n_2=n$

$$=\left(\frac{1}{4}\right)^{100}\left(\sum_{n}{}^{100}C_{n}\right)^2$$

Correct option is (b)

JNU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. Three dics with faces marked 1, 2, ..., 6 are thrown together. Assuming that they are unbiased, what is the probability that the sum of the numbers that turn up is 15? **[JNU 2010]**

(a) 1/108 (b) 3/108 (c) 5/108 (d) 7/108

Soln: $x(s) = 6 \times 6 \times 6$

$E =$ sum is 15

$= \{(3,6,6), (6,3,6), (6,6,3), (4,6,5), (4,5,6), (5,6,4), (6,5,4), (5,4,6), (6,4,5), (5,5,5)\}$

Hence, required probability, $\frac{10}{6\times6\times6} = \frac{5}{108}$.

Correct option is (c)

HCU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. If one person speaks the truth in 70% of the cases and another in 80% of the cases, then the probability that they will contradict each other in describing a single event is

(a) 0.36 (b) 0.38 (c) 0.40 (d) 0.42 **[H.C.U.-2012]**

Soln: $P(A)$ = probability that 1st person speaks truth = 70/100.

$P(B)$ = probability that 2nd person speaks truth = 80/100.

Hence, required probability = (1st speaks truth and 2nd lies) + (2nd speaks truth and 1st lies).

$$= \frac{70}{100} \times \frac{20}{100} + \frac{80}{100} \times \frac{30}{100} = \frac{14+24}{100} = 0.38.$$

Correct option is (b)

Question 2. A number is choosen at random from the set of integers {1, 2, ..., 1000}. What is the probability that it is divisible by 3 or 5? [i.e., either 3 or 5 or both]

(a) 0.333 (b) 0.2 (c) 0.1 (d) 0.467 **[H.C.U.-2013]**

Soln: A = event that choosen number is divisible by 3.

$= \{3, 6, 9, 12, 15, ..., 999\}$

$n(A) = 333$

B = even that choosen number is divisible by 5.

$= \{5, 10, 15, 20, ..., 1000\}$

$n(B) = 200$

Numbers divisible by both 3 and 5.

$\{15, 30, 45, ..., 990\}$

$n(A \cap B) = 66.$

Therefore, probability that choosen number is divisible by either 3 or 5.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{333}{1000} + \frac{200}{1000} - \frac{66}{1000} = 0.467.$$

Question 3. If a bag contains blue, green and red marbles, all but 15 are blue, all but 13 are red and all but 12 are green. How many are red?

(a) 13 (b) 7 (c) 8 (d) 20 **[H.C.U.-2014]**

Soln: Let number of blue marbles = X
number of green marbles = Y
number of red marbles = Z
Since, all but 15 are blue, $Y + Z = 15$ ... (1)
All but 13 are red, $X + Y = 13$ ... (2)
All but 12 are green, $X + Z = 12$ ... (3)
Solving Eqs. (1), (2) and (3) for Z,
Therefore, $Z = 7$.
Correct option is (b)

BHU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The probability of drawing two aces in succession from a pack of 52 cards is

(a) $\frac{2}{26}$ (b) $\frac{2}{13}$ (c) $\frac{1}{221}$ (d) $\frac{1}{169}$ **[B.H.U-2010]**

Soln: Required probability $= \frac{{}^4C_1}{52} \times \frac{{}^3C_1}{52} = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$.

Correct option is (c)

Question 2. Two non-interacting particles are distributed in three distinct states. Let P_c be the probability for both of them in the same state in case particles are distinguishable and P_b the probability for them to be in the same state in case they are indistinguishable bosons. The ratio P_c/P_b is

(a) $\frac{3}{2}$ (b) 1 (c) $\frac{2}{3}$ (d) $\frac{1}{3}$ **[B.H.U-2010]**

Soln: If particles are distinguishable (a, b)

ε_1	ε_2	ε_3
ab		
	ab	
		ab
a	b	
	a	b
b	a	
	b	a
a		b
b		a

Hence, probability for both of them to be in the same state $= P_c = \frac{3}{9} = \frac{1}{3}$.

If particles are indistinguishable (a, a)

ε_1	ε_2	ε_3
aa		
	aa	
		aa
a	a	
	a	a
a		a

Hence, probability for both them to be in the same state = $P_b = \frac{3}{6} = \frac{1}{2}$.

Therefore, $\frac{P_c}{P_b} = \frac{1}{3} \times \frac{2}{1} = \frac{2}{3}$.

Correct option is (c)

Question 3. A card is drawn a pack containing 52 cards with 4 aces and another card is drawn from a pack of 48 cards with 8 aces. What is the probability that both are aces? **[B.H.U-2012]**

(a) $\frac{4}{52}$ (b) $\frac{8}{48}$ (c) $\frac{32}{52 \times 48}$ (d) $\frac{1}{78}$.

Soln: Probability of drawing an ace in 1st case, $P_1 = \frac{4}{52}$, and probability of drawing an ace in 2nd case, $P_2 = \frac{8}{48}$.

Then probability that both cards are ace is $= \frac{4}{52} \times \frac{8}{46} = \frac{1}{78}$.

Correct option is (d)

Question 4. In a coded telegram, the letters are arranged in groups of 5 letters called 'words'. How many different 'such words are there which contains each letter at most once? **[B.H.U-2012]**

(a) $\frac{26!}{\{26-5\}!}$ (b) $\frac{26!}{5!}$ (c) $\frac{26!}{5!\{26-5\}!}$ (d) $\{26-5\}!$

Soln: 5 letters can be chosen from 26 letters in ${}^{26}C_5$ way.

Also, all five letters can be arranged among themselves in 5! way.

Hence, total number of words

${}^{26}C_5 \times 5! = \frac{26!}{(26-5)!\,5!} \times 5! = \frac{26!}{(26-5)!}$.

Correct option is (a)

Question 5. In a random walk problem, if the probability that a particle is found between x to $x+dx$ is given as $P(x)=e^{-\pi x^2}$, the mean $x(\bar{x})$ is **[B.H.U-2013]**

(a) 0 (b) 1 (c) π (d) none of these

$$\langle x\rangle = \frac{\int_{-\infty}^{\infty} x\,p(x)\,dx}{\int_{-\infty}^{\infty} p(x)\,dx}$$

Soln: Here, $\int_{-\infty}^{+\infty} p(x)\,dx = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = \sqrt{\frac{\pi}{\pi}} = 1$

$$\int_{-\infty}^{\infty} x\,p(x)\,dx = \int_{-\infty}^{\infty} x e^{-\pi x^2}\,dx = 0 \qquad (\because \text{odd function})$$

Therefore, $\langle x\rangle = 0$.

Correct option is (a)

Question 6. If the probability in a total number of N steps making n_1 steps to the right and n_2 steps to the left is described by a binomial distribution, the mean number (n_1) is proportional to **[B.H.U-2015]**

(a) N (b) N^{-1} (c) $N^{1/2}$ (d) none of these

Soln: $n_1 + n_2 = N$

$p(n_1) = {}^{N}C_{n_1}(p)^{n_1}(q)^{n_2}$, where $p+q=1$

$$\langle n_1\rangle = \sum n_1\,p(n_1) = \sum_{n_1,n_2} n_1 \frac{N!}{n_1!n_2!}(p)^{n_1}(q)^{n_2}$$

$$= p\frac{\partial}{\partial p}\sum_{n_1,n_2}\frac{N!}{n_1 n_2}(p)^{n_1}(q)^{n_2} = p\frac{\partial}{\partial p}(p+q)^N$$

$$= Np(p+q)^{N-1} = Np.$$

Correct option is (a)

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Chapter 9

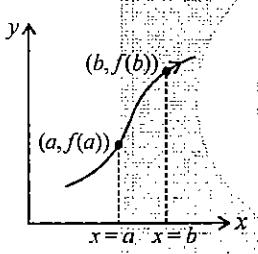
CALCULUS OF SINGLE & MULTIPLE VARIABLES

9.1 Monotonic Function

A monotonic function is a function which is either entirely non-increasing or non-decreasing. The first derivative of the monotonic function (which need not be continuous) does not change sign.

(1) A function is called monotonically increasing (also increasing or non-decreasing), if for all $x = a$ and b such that $a \le b$, one has $f(a) \le f(b)$, so 'f' preservers the order.

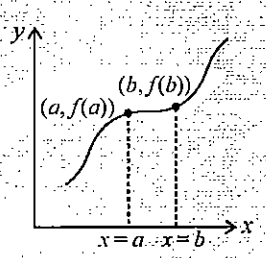
Example: (i)



$f(a) < f(b)$

(Strictly increasing)

(ii)

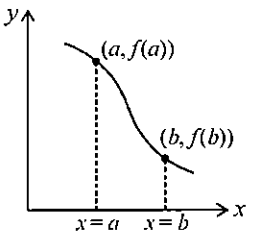


$f(a) \le f(b)$

(Only increasing)

(2) A function is called monotonically decreasing (also decreasing or non-increasing), if for all $x = a$ and b, such that $a \le b$, one has $f(a) \ge f(b)$, so, 'f' reverse the order.

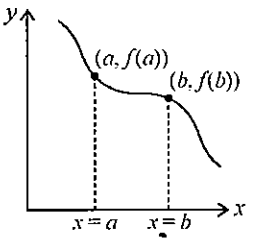
Example: (i)



$f(a) > f(b)$

(Strictly decreasing)

(ii)



$f(a) \ge f(b)$

(Only decreasing)

9.2 Test for increasing and decreasing nature of a function at any point:

(i) If $f'(a) > 0$, then $f(x)$ is increasing at $x = a$.

(ii) If $f'(a) < 0$, then $f(x)$ is decreasing at $x = a$.

(iii) If $f'(a) = 0$, then check the sign of $f'(a^+)$ and $f'(a^-)$.

(a) If $f'(a^+) > 0$ and $f'(a^-) > 0$, then increasing.

(b) If $f'(a^+) < 0$ and $f'(a^-) < 0$, then decreasing.

(c) Otherwise neither increasing nor decreasing.

Example 1. Let $f(x) = x^3 - 3x^2 + 3x + y$. Examine the nature of function at $x = 0$ and 1.

Soln: $f(x) = x^3 - 3x^2 + 3x + 4 \Rightarrow f'(x) = 3x^2 - 6x + 3$

At $x = 0$, $f'(0) = 3 > 0$. Hence $f(x)$ is increasing at $x = 0$.

At $x = 1$, $f'(1) = 3 - 6 + 3 = 0$. Hence, let's check $f'(1^+)$ and $f'(1^-)$.

Clearly, $f'(1^+) = f(x = 1.001) > 0$, and $f'(1^-) = f(x = 0.999) > 0$.

Hence $f(x)$ is increasing at $x = 1$.

9.3 Steps to find the intervals for increasing and decreasing behaviour of a function:

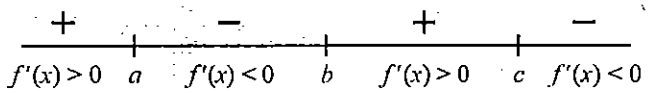
Let given function is $y = f(x)$.

(i) Find $f'(x)$ and equate it to zero. i.e., $f'(x) = 0$

(ii) Find the roots of $f'(x) = 0$ and Let it is a, b, c etc.

(iii) Now use sign scheme for $f'(x)$.

Suppose it is like this



Then, intervals for increasing $(f'(x) > 0)$ is $x \in (-\infty, a) \cup (b, c)$

and, intervals for decreasing $(f'(x) < 0)$ is $x \in (a, b) \cup (c, \infty)$

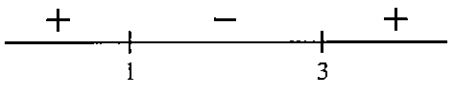
Example 2. Find the intervals of monotonicity for the following function

$$f(x) = -x^3 + 6x^2 - 9x - 2$$

Soln: (i) $f(x) = -x^3 + 6x^2 - 9x - 2 \Rightarrow f'(x) = -3x^2 + 12x - 9$

Now, $f'(x) = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0 \Rightarrow x = 1, 3$

Now, using sign scheme for $f'(x)$ (quadratic).



Hence, intervals for increasing is $x \in (-\infty, 1) \cup (3, \infty)$ and intervals for decreasing is $x \in (1, 3)$.

9.4 Steps to find points of maxima and minima of a function containing single variable:

Method 1:

Let given function is $y = f(x)$

(i) Find roots for $f'(x) = 0$ and Let roots are $x = a, b$ etc.

(ii) Use sign scheme. The point(s) at which $f'(x)$ changes sign from (+) to (–) is/are the points of maxima (also called local maxima) and the point(s) at which $f'(x)$ changes sign from (–) to (+) is/are the points of minima (also called local minima).

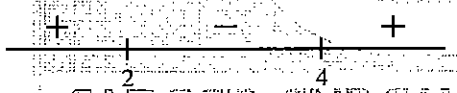
Example 3. Find the maxima and minima of the function $f(x) = x^3 - 9x^2 + 24x$.

Soln: Here, $f'(x) = 3x^3 - 18x^2 + 24$

Applying $f'(x) = 0 \Rightarrow 3x^2 - 18x + 24 = 0$

$\Rightarrow x^2 - 6x + 8 = 0 \Rightarrow (x-4)(x-2) = 0 \Rightarrow x = 4, 2$

Now, applying sign scheme



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At $x = 2$, sign changes from (+) to (–), hence $x = 2$ is a point of local maxima.

At $x = 4$, sign changes from (–) to (+), hence $x = 4$ is a point of local minima.

Method 2:

Let given function is $y = f(x)$

(i) Find roots of $f'(x) = 0$ and Let roots are $x = a, b$ etc.

(ii) Now find $f''(x)$, if $f''(a) > 0$, $x = a$ is a point of local minima, and if $f''(a) < 0$, $x = a$ is a point of local maxima.

Example 4. Find the maxima and minima of the function $f(x) = \dfrac{x+1}{\sqrt{5x^2+35}}$

Soln: Here, $f'(x) = \dfrac{(5x^2+35)^{1/2}(1) - (x+1) \times \dfrac{10x}{2(5x^2+35)^{1/2}}}{(5x^2+35)}$

Applying $f'(x)=0 \Rightarrow 2(5x^2+35)-10x(x+1)=0 \Rightarrow x=7$

Now, $f''(x=7)=-10<0$

Hence, $x=7$ is a point of local maxima.

Example 5. Let $F(x)=\int_0^x \left(t^2-3t+2\right)dt$, then F has

(a) a local maximum at $x=1$ and a local minimum at $x=2$

(b) a local minimum at $x=1$ and a local maximum at $x=2$

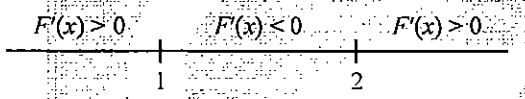
(c) local maxima at $x=1$ and $x=2$

(d) local minima at $x=1$ and $x=2$

Soln: $F(x)=\int_0^x \left(t^2-3t+2\right)dt \Rightarrow F'(x)=x^2-3x+2$

Hence for maxima and minima, $F'(x)=0 \Rightarrow x^2-3x+2=0 \Rightarrow x=1,2$

Now, use sign scheme for $F'(x)$



Hence, $x=1$ is a point of local maxima and $x=2$ is a point of local minima.

Correct option is (a)

Example 6. Suppose that f is a continuous and differentiable on $[a, b]$. If $f'(x) \le 0$ on $[a, c)$ and $f'(x) \ge 0$ on $(c, b]$, $a<c<b$, then on $[a, b]$.

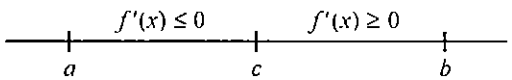
(a) $f(x)$ is never less than $f(c)$ (b) $f(x)$ is always less than $f(c)$

(c) $f(x)$ is always less than $f(a)$ (d) $f(x)$ is always greater than $f(b)$

Soln: Since, $f'(x) \le 0$ on $[a, c]$ and $f'(x) \ge 0$ on $(c, b]$

Hence, $f'(c)=0$

Now, use sign scheme



Hence, point $x=c$ is a point of local minima and $f(c)$ is minimum value of $f(x)$ on $[a, b]$. Hence, $f(x)$ is always greater than $f(c)$.

Correct option is (a)

Example 7. Let $f(x)=2x^3-x^2+2x-5$. Consider the following statements about the roots of $f(x)=0$.

A: At least one root is positive.

B: At least one root is negative.

C: There is a root between $x=1$ and .

Which one of the following is TRUE?

(a) A, B and C are valid statements.
(b) A and B are valid statements, but C is not a valid statement.
(c) A and C are valid statements, but B is not a valid statement.
(d) A is valid statement, but B and C are not valid statements.

Soln: $f(x) = 2x^3 - x^2 + 2x - 5$

Since, there are three sign change in the expression of the function, thus there will be at least three positive roots of $f(x)$ will exist, i.e., it will have no negative roots.

Also, $f(1) = -2$ and $f(2) = 11$

Thus, $f(x)$ will have at least one root between $x = 1$ and $x = 2$.

So, A and C are valid statements.

Correct option is (c)

9.5 Steps to find points of maxima and minima of a function containing multiple variables:

Let given function is $z = f(x, y)$.

(i) Find roots for $f_x(x, y) = \frac{\partial f}{\partial x} = 0$ and $f_y(x, y) = \frac{\partial f}{\partial y} = 0$. Let roots are $x = x_0$, $y = y_0$ and this points are known as critical points.

(ii) Find $D(x_0\ y_0) = f_{xx}(x_0\ y_0) f_{yy}(x_0\ y_0) - f_{xy}(x_0\ y_0)^2$,

$$\left(\text{where } f_{xx} = \frac{\partial^2 f(x, y)}{\partial x^2},\ f_{yy} = \frac{\partial^2 f(x, y)}{\partial y^2} \text{ and } f_{xy} = \frac{\partial^2 f(x, y)}{\partial x \partial y} \right)$$

(iii) (a) If $D > 0$ and $f_{xx}(x_0\ y_0) < 0$, there is a local maximum at $(x_0\ y_0)$

(b) If $D > 0$ and $f_{xx}(x_0\ y_0) > 0$, there is a local minimum at $(x_0\ y_0)$

(c) If $D < 0$, there is neither a maximum nor a minimum at $(x_0\ y_0)$

(d) If $D = 0$, the test fails.

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Example 8. Find all local maxima and minima for $f(x, y) = x^2 - y^2$.

Soln: Here, $f_x = 2x$, $f_y = -2y$, $f_{xx} = 2$, $f_{yy} = -2$, $f_{xy} = 0$

Applying $f_x = 0$, $f_y = 0$, we get $x = 0$ and $y = 0$. So, the critical ponit of the function is (0, 0).

Here, $D(0, 0) = f_{xx}(0, 0)\ f_{yy}(0, 0) - f_{xy}(0, 0)^2 = (2 \times -2) - 0 = -4 < 0$.

So, there is neither a maximum nor minimum at (0, 0), and so there are no local maxima or minima of the function.

Example 9. Find all local maximum and minimum points of $f_x(x, y) = x^2 + 4y^2 - 2x + 8y - 1$.

Soln: Here, $f_x = 2x - 2$, $f_y = 8y + 8$, $f_{xx} = 2$, $f_{yy} = 8$; $f_{xy} = 0$

Applying $f_x = 0$, $f_y = 0$, we get $x = 1$ and $y = -1$. So, the critical ponit of the function is (1, -1).

Here, $D(1, -1) = f_{xx}(1, -1)\ f_{yy}(1, -1) - f_{xy}(1, -1)^2 = 2 \times 8 - 0 = 16 > 0$ and $f_{xx}(1, -1) > 0$

So, there is a local minima at $(1, -1)$.

Example 10. The function $f(x, y) = x^3 + 3xy^2 - 4y^2 - 4y^3 - 15x$ has a local

(a) minima at $(-\sqrt{5}, 0)$ (b) minima at $(\sqrt{5}, \sqrt{5})$

(c) maxima at $(\sqrt{5}, 0)$ (d) maxima at $(-\sqrt{5}, 0)$

Soln: $f(x, y) = x^3 + 3xy^2 - 4y^2 - 4y^3 - 15x$

$$f_x = \frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 15 = 0 \Rightarrow x^2 + y^2 = 5 \quad ...(1)$$

$$f_y = \frac{\partial f}{\partial y} = 6xy - 12y^2 = 0 \Rightarrow 6y(x - 2y) = 0 \Rightarrow y = 0 \text{ and } y = x/2$$

If $y = 0$, from equation (1), $x = \pm\sqrt{5}$ and if $y = \frac{x}{2}$, from equation (1), $x^2 + \frac{x^2}{4} = 5 \Rightarrow x = \pm 2$

Hence, critical points of the function are $(\pm\sqrt{5}, 0)$ and $(\pm 2, \pm 1)$

We have to check only at $(\pm\sqrt{5}, 0)$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 9x \quad ; f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 6y \quad ; f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6x - 24y$$

Here, $D(\sqrt{5}, 0) = (f_{xx} f_{yy} - f_{xy}^2)\big|_{(\sqrt{5}, 0)} > 0$ and $f_{xx}(\sqrt{5}, 0) < 0$. Therefore, the function has a local minima at $(\sqrt{5}, 0)$.

Here, $D(-\sqrt{5}, 0) = (f_{xx} f_{yy} - f_{xy}^2)\big|_{(-\sqrt{5}, 0)} > 0$ and $f_{xx}(-\sqrt{5}, 0) < 0$. Therefore, the function has a local maxima at $(-\sqrt{5}, 0)$.

Correct option is (d)

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Example 11. Let $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$. Choose the correct statement about $f(x, y)$.

(a) $f(x, y)$ has neither a maximum nor a minimum at (0, 0)

(b) $f(x, y)$ has a local maxima at (0, 0)

(c) $f(x, y)$ has a local minima at (0, 0)

(d) None of the above are correct

Soln: $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$

$$f_x = \frac{\partial f}{\partial x} = 2x - 2y + 3x^2 + 5x^4 \text{ and } f_y = \frac{\partial f}{\partial y} = -2x + 2y - 3y^2$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2 + 6x + 20x^3, \; f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2 - 6y \text{ and } \frac{\partial^2 f}{\partial x \partial y} = -2$$

Applying $f_x = 0$, $f_y = 0$, we get $x = 0$ and $y = 0$. So, the critical ponit of the function is (0, 0).

Now, $D(0,0) = \left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{(0,0)} = 0$

Hence, the point (0, 0) is a doubtful point, hence needs further investigation.

We can write $f(x,y) = (x-y)^2 + (x-y)(x^2+xy+y^2) + x^5$

We have $f(0,0) = 0$

In the neighbourhood of the point (0, 0), where $x = y$, we have $f(x,y) = x^5$, which is positive when $x > 0$ and is negative when $x < 0$.

Thus, in the neighbourhood of the point (0, 0), there are points at which $f(x,y)$ takes values less than its value at the point (0, 0) and there are points at which $f(x,y)$ takes values greater than its value at (0, 0).

Hence, $f(x,y)$ can not have a maximum or a minimum at the point (0, 0).

Correct option is (a)

9.5 Multiple Integral

A multiple integral is a generalization of the usual integral in one dimension to functions of multiple variables in higher-dimensional spaces. The most common multiple integrals are double and triple, involving two or three variables, respectively.

Double Integral

The double integral is written as $\iint f(x,y)\,dx\,dy$.

Here, $f(x,y)$ can be function of x and y both or function of x or y or it can be constant.

To integrate,

(i) One should treat y as a constant and perform the integration with respect to x,

(ii) After that, perform the integration with respect to y.

The limit are of the form

$$\int_{y=c}^{y=d}\int_{x=f_1(y)}^{x=f_2(y)} f(x,y)\,dx\,dy$$

Similarly, for $\iint f(x,y)\,dy\,dx$, $f(x,y)$ is first integrated with respect to y and then it is integrated with respect to x.

The limit are the form

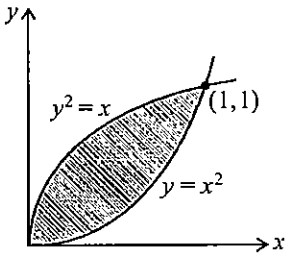
$$\int_{x=c}^{x=d}\int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\,dy\,dx.$$

Example 12. Compute the double integral $I = \int_0^1\int_0^1 \frac{x^2+y^2}{2}\,dx\,dy$.

Soln: $I=\frac{1}{2}\int_0^1\left(\frac{x^3}{3}+y^2x\right)_0^1 dy=\frac{1}{2}\int_0^1\left(\frac{1}{3}+y^2\right)_0^1 dy=\frac{1}{2}\left(\frac{1}{3}y+\frac{y^3}{3}\right)_0^1=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}\right)=\frac{1}{3}.$

Example 13. Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region bounded by $y=x^2$, $y^2=x$.

Soln: The region of integration is bounded by $C_1: y=x^2$ and $C_2: y^2=x$



$$I=\iint (x^2+y^2)\,dx\,dy=\int_{y=0}^{1}\int_{x=y^2}^{\sqrt{y}} (x^2+y^2)\,dx\,dy.$$

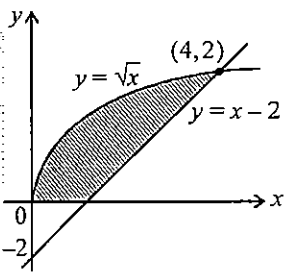
$$I=\int_{y=0}^{1}\left(\frac{x^3}{3}+xy^2\right)_{y^2}^{\sqrt{y}} dy=\int_{y=0}^{1}\left\{\left(\frac{1}{3}y^{3/2}+\sqrt{y}y^2\right)-\left(\frac{y^6}{3}+y^4\right)\right\}dy$$

$$=\int_{y=0}^{1}\left(\frac{1}{3}y^{3/2}+y^{5/2}-\frac{y^6}{3}-y^4\right)dy=\left(\frac{1}{3}\frac{y^{5/2}}{5/2}+\frac{y^{7/2}}{7/2}-\frac{y^7}{3\times 7}-\frac{y^5}{5}\right)_0^1$$

$$=\frac{2}{15}+\frac{2}{7}-\frac{1}{21}-\frac{1}{5}=\frac{6}{35}.$$

Example 14. Find the area of the region in the 1st quadrant that is bounded by $y=\sqrt{x}$, $y=x-2$ and the x-axis.

Soln: Region is bounded by the curves, $C_1: y=\sqrt{x}$ and $C_2: y=x-2$.



$$\text{Required area}=\int_{y=0}^{2}\int_{x=y^2}^{x=y+2} dx\,dy=\int_{y=0}^{2}(x)_{y^2}^{y+2}dy=\int_0^2 (y+2-y^2)\,dy$$

$$=\left(\frac{y^2}{2}+2y-\frac{y^3}{3}\right)_0^2=2+4-\frac{8}{3}=\frac{10}{3}.$$

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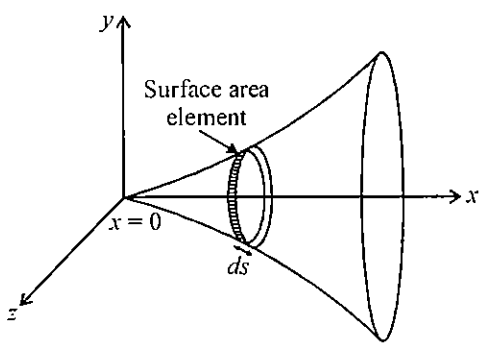
Example 15. Consider the curve $y=x^2$ from $x=0$ to $x=1$. Now, rotate the area about the x-axis to form a volume and surface of revolution. Then, calculate the area of the curved surface from $x=0$ to $x=1$.

Soln: We find the area of the surface of revolution by using as element the curved surface of a thin slab as shown in figure. This is a strip of circumference $2\pi y$ and width ds.

Here, $ds=$ arc length $=\left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2} dx$

Hence, total area of revolution,

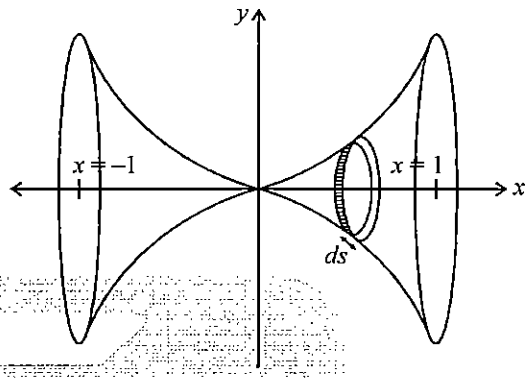
$$A=\int_{x=0}^{1} 2\pi y\,ds=\int_0^1 2\pi x^2\sqrt{1+4x^2}\,dx$$



$$= \frac{2\pi}{64}\left[18\sqrt{5} - \sinh^{-1}(2)\right].$$

Example 16. Find the area of the surface obtained by rotating about the x-axis the arc of the curve $y = \frac{x^3}{3}$, from $x = -1$ to $x = 1$.

Soln: $A = 2\int 2\pi y\, ds$



$$A = 4\pi \int_0^1 \frac{x^3}{3}\sqrt{1+x^4}\, dx$$

$$1 + x^4 = t^2,\ 4x^3 dx = 2t\, dt$$

$$A = \frac{4\pi}{3}\int_1^{\sqrt{2}} t \times \frac{t\, dt}{2}$$

$$= \frac{4\pi}{3} \times \frac{1}{2}\left(\frac{t^3}{3}\right)_1^{\sqrt{2}} \quad = \frac{2\pi}{9}\left(2\sqrt{2} - 1\right)$$

Triple Integral

The triple integral is written as $\int\int\int f(x, y, z)\, dx\, dy\, dz$.

Here, $f(x, y, z)$ can be function of x, y and z both or function of x or y or z or it can be a constant.
To integrate,

(i) One should treat y, z as constants and perform the integration with respect to x,

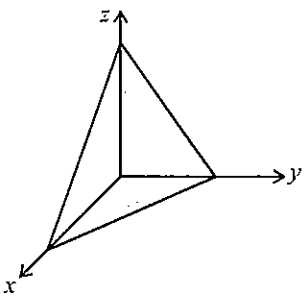
(ii) After that, perform the integration with respect to y, treating z as a constant.

(iii) At last, perform the integration with respect to z.

The limit are of the form

$$\int_{z=c}^{z=d}\int_{y=f_3(z)}^{y=f_4(z)}\int_{x=f_1(y,z)}^{x=f_2(y,z)} f(x, y, z)\, dx\, dy\, dz$$

Example 17. Evaluate $\iiint_E 2x\, dV$, where E is the region under the plane $2x + 3y + z = 6$ that lies in the first octant.



Soln: $V = \iiint 2x\,dV = \iint 2x\,dx\,dy\,dz$

Plane is $2x + 3y + z = 6$

Hence, z will vary from 0 to $(6 - 3y - 2x)$

x will vary from 0 to $\left(-\frac{3}{2}y + 3\right)$

y will vary from 0 to 2

Therefore, $V = \iiint_{z=0}^{6-3y-2x} 2x\,dz\,dy\,dx$

$$= \int_{y=0}^{2} \int_{x=0}^{-\frac{3}{2}y+3} 2x(6 - 2x - 3y)\,dx\,dy = 9$$

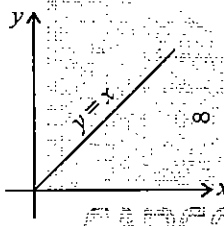
Example 18. The value of the integral $\int_0^{\infty} \int_{y=0}^{x} x e^{-x^2/y}\,dy\,dx$ is

(a) 0 (b) 1/2 (c) 4 (d) 1

Soln: $I = \int_0^{\infty} \int_{y=0}^{x} x e^{-x^2/y}\,dy\,dx$

Domain of integration is bounded by $y = 0,\ y = x,\ x = 0\ ;\ x = \infty$

Changing the order



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Therefore, $I = \int_0^{\infty} \int_{y}^{\infty} x e^{-x^2/y}\,dx\,dy = -\int_0^{\infty} \left(\frac{y}{2} e^{-x^2/y}\right)_{y}^{\infty} dy$

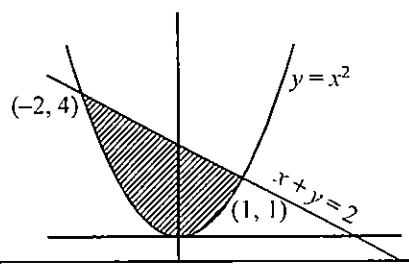
$$= \frac{1}{2} \int_0^{\infty} y e^{-y}\,dy = \frac{1}{2}\left[-y e^{-y} - e^{-y}\right]_0^{\infty} = \frac{1}{2}$$

Correct option is (b)

Example 19. The area of the region enclosed by the curve $y = x^2$ and the straight line $x + y = 2$ is

(a) 3 (b) 27/2 (c) 9/2 (d) 9

Soln: Required region is shown in figure :



Solving the two curves $y = x^2$ and $x + y = 2$, we get the points of intersection as $(-2, 4)$ and $(1, 1)$.

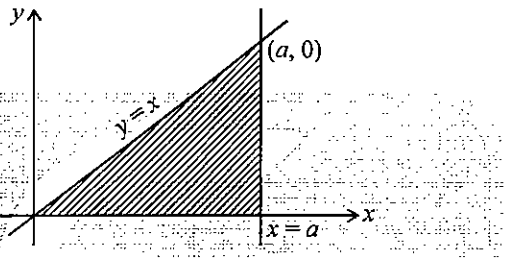
Hence, required area $= \int_{x=-2}^{1} \int_{y=x^2}^{y=2-x} dy\,dx = \int_{-2}^{1} (2 - x - x^2)\,dx = \frac{9}{2}$.

Correct option is (c)

Example 20. Let $F : R \to R$ be a continuous function and $a > 0$, then the integral $\int_0^a \left[\int_0^x F(y)\,dy \right] dx$ equals

(a) $\int_0^a y F(y)\,dy$ (b) $\int_0^a (a - y) F(y)\,dy$ (c) $\int_0^a (y - a) F(y)\,dy$ (d) $\int_0^a y F(y)\,dy$

Soln: Domain of integration is bounded by $y = 0$, $y = x$, $x = 0$, $x = a$



Hence, changing the order

$$I = \int_0^a \left[\int_0^x F(y)\,dy \right] dx = \int_{y=0}^{a} \left[\int_{x=y}^{a} F(y)\,dx \right] dy = \int_{y=0}^{a} (a - y) F(y)\,dy.$$

Correct option is (b)

CAREER ENDEAVOUR

Chapter 10

PARTIAL DIFFERENTIATION

Introduction:

Let z denote the function $f(x,y)$ which posseses one definite value for every pair of values of x and y, and moreover if y is kept constant, $z = f(x,y)$ varies because of x, and similarly if x is kept constant, z varies because of the variation in a single variable y. A similar definition can be given for functions of more than two variables, i.e., $u = f(x,y,z)$ denotes a function of three variables x, y, z.

Partial derivatives of $f(x,y)$:

The ordinary derivative of $f(x,y)$ with respect to x keeping the other variable constant is called the partial derivtive of $f(x,y)$ with respect to x. Similarly ordinary derivative of $f(x,y)$ with respect to y keeping the other variable x constant is called the partial derivative of $f(x,y)$ with respect to y.

The partial derivative of $f(x,y)$ with respect to x and y are generally denoted by $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$. Thus by definition we write

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \quad \text{and} \quad \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,k+y) - f(x,y)}{k}$$

when these limits exist.

At a particular point (a,b), these partial derivatives are often denoted by $f_x(a,b)$ and $f_y(a,b)$. Thus,

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}; \; f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

Again $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}; \; f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$

provided the limit exist.

Example 1. Find the first and second order partial derivatives of $x^3 + y^3 + 3axy$

Soln: Let $f(x,y) = x^3 + y^3 + 3axy$, then $\frac{\partial f}{\partial x} = 3x^2 + 3ay$, $\frac{\partial f}{\partial y} = 3y^2 + 3ax$

$$\frac{\partial^2 f}{\partial y \partial x} = 3a, \quad \frac{\partial^2 f}{\partial x \partial y} = 3a; \; \frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = 6y;$$

Example 2. If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = \left(1 + 3xyz + x^2y^2z^2\right)e^{xyz}$

Soln: We have $u = e^{xyz}$, $\Rightarrow \frac{\partial u}{\partial x} = e^{xyz}.yz \Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(yze^{xyz}\right) = z\left[ye^{xyz}.xz + e^{xyz}\right]$

$$\Rightarrow \frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{\partial}{\partial z}\left[xyz^2e^{xyz} + ze^{xyz}\right] = xy\left[z^2e^{xyz}.xy + e^{xyz}.2z\right] + \left[ze^{xyz}.xy + e^{xyz}\right]$$

$$= e^{xyz}\left(1 + 3xyz + x^2y^2z^2\right)$$

Example 3: If $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$ show that

(a) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = -\frac{9}{(x+y+z)^2}$ and (b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{z}{(x+y+z)^2}$

Soln: (a) $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right) \Rightarrow \frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} \text{ and } \frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\left(x^2 + y^2 + z^2 - xy - yz - zx\right)}{x^3 + y^3 + z^3 - 3xyz} = \frac{3}{x+y+z}$$

Now, $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}$$

(b) $\frac{\partial^2 u}{\partial x^2} = \frac{6x\left(x^3 + y^3 + z^3 - 3xyz\right) - \left(3x^2 - 3yz\right)\left(3x^2 - 3yz\right)}{\left(x^3 + y^3 + z^3 - 3xyz\right)^2} = -\frac{3\left(x^4 - 2xy^3 - 2xz^3 + 3y^2z^2\right)}{\left(x^3 + y^3 + z^3 - 3xyz\right)^2}$

Similarly, $\frac{\partial^2 u}{\partial y^2} = -\frac{3\left(y^4 - 2yz^3 - 2yx^3 + 3z^2x^2\right)}{\left(x^3 + y^3 + z^3 - 3xyz\right)^2}$ and $\frac{\partial^2 u}{\partial z^2} = -\frac{3\left(z^4 - 2zx^3 - 2zy^3 + 3x^2y^2\right)}{\left(x^3 + y^3 + z^3 - 3xyz\right)^2}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3\left(x^2 + y^2 + z^2 - xy - yz - zx\right)^2}{\left(x^3 + y^3 + z^3 - 3xyz\right)^2} = -\frac{3}{(x+y+z)^2}$$

Example 4. If $x^x.y^y.z^z = c$ show that at $x = y = z$, $\dfrac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$

Soln: Given that, $x^x y^y z^z = c$

Taking log of both sides, we have, $x \log x + y \log y + z \log z = \log c$

Differentiating partially w.r.t. x, we get,

$$x.\frac{1}{x} + \log x + \left(z.\frac{1}{z} + \log z\right)\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)}$$

Similarly, we have, $\dfrac{\partial z}{\partial y} = -\dfrac{(1+\log y)}{(1+\log z)}$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{(1+\log x)}{(1+\log z)^2}.\frac{1}{z}\frac{\partial z}{\partial y} = -\frac{(1+\log x)}{(1+\log z)^2}.\frac{1}{z}.\frac{(1+\log y)}{(1+\log z)} = -\frac{(1+\log x)^2}{(1+\log x)^3}.\frac{1}{x} \text{ (since } x = y = z)$$

$$= -\frac{1}{x(1+\log x)} = -\frac{1}{x \log ex} = -(x \log ex)^{-1}$$

Example 5: If $z = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, prove that $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{x^2 - y^2}{x^2 + y^2}$

Soln: We have, $\dfrac{\partial z}{\partial x} = 2x \tan^{-1}\dfrac{y}{x} + \dfrac{x^2}{1+(y/x)^2}\left(-\dfrac{y}{x^2}\right) - \dfrac{y^2}{1+(x/y)^2}\left(\dfrac{1}{y}\right)$

$$= 2x \tan^{-1}(y/x) - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2} = 2x \tan^{-1}(y/x) - \frac{y(x^2+y^2)}{(x^2+y^2)} = 2x \tan^{-1}(y/x) - y$$

Now, $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial y}\left\{2x \tan^{-1}\dfrac{y}{x} - y\right\} = \dfrac{2x}{1+(y/x)^2}\left(\dfrac{1}{x}\right) - 1 = \dfrac{2x^2}{x^2+y^2} - 1 = \dfrac{x^2-y^2}{x^2+y^2}$

Example 6. If $u = f(r)$, where $r^2 = x^2 + y^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f'(r) + \frac{1}{r} f'(r)$$

Soln: We have, $2r\dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ and $2r\dfrac{\partial r}{\partial y} = 2y$ or $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

Now, $u = f(r) \Rightarrow \dfrac{\partial u}{\partial x} = f'(r).\dfrac{\partial r}{\partial x} = f'(r).\dfrac{x}{r}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{f'(r)x}{r}\right] = \frac{r\left\{f'(r).1 + x f''(r)\left(\dfrac{\partial r}{\partial x}\right)\right\} - x f'\left(\dfrac{\partial r}{\partial x}\right)}{r} = \frac{1}{r^2}\left[r f'(r) + x^2 f''(r) - \frac{x^2}{r} f'(r)\right]$$

Similary, $\frac{\partial^2 u}{\partial y^2}=\frac{1}{r^2}\left[r f'(r)+y^2 f''(r)-\frac{y^2}{r}f'(r)\right]$

Hence, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial y^2}=\frac{1}{r^2}\left[2rf'(r)+\left(x^2+y^2\right)f''(r)-\frac{\left(x^2+y^2\right)}{r}f'(r)\right]$

$$=\frac{1}{r^2}\left[2rf'(r)+r^2 f''(r)-rf'(r)\right]=\frac{1}{r}f'(r)+f''(r)$$

Example 7. If $\theta=t^n e^{-r^2/4t}$, find what value of n makes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2.\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}$$

Soln: We have, $\theta=t^n e^{-r^2/4t}\Rightarrow\frac{\partial\theta}{\partial r}=t^n e^{-r^2/4t}\left(-\frac{2r}{4t}\right)=-\frac{1}{2}rt^{n-1}e^{-r^2/4t}$

$$\Rightarrow\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial}{\partial r}\left[-\frac{1}{2}r^3t^{n-1}e^{-r^2/4t}\right]=-\frac{1}{2}t^{n-1}\left[r^3.e^{-r^2/4t}\left(-\frac{2r}{4t}\right)+3r^2.e^{-r^2/4t}\right]$$

$$\Rightarrow\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{1}{4}t^{n-2}\left[r^2-6t\right]e^{-r^2/4t}$$

$$\frac{\partial\theta}{\partial t}=t^n.e^{-r^2/4t}\frac{r^2}{4t^2}+e^{-r^2/4t}.nt^{n-1}=\frac{1}{4}t^{n-2}e^{-r^2/4t}\left(r^2+4nt\right)$$

Now to satisfy the condition given in the question, we have

$$\frac{1}{4}t^{n-2}\left[r^2-6t\right]e^{-r^2/4t}=\frac{1}{4}t^{n-2}\left(r^2+4nt\right)\Rightarrow r^2-6t=r^2+4nt\Rightarrow n=-3/2$$

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Homogeneous Functions:

An expression of the type $a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+a_3x^{n-3}y^3+.....a_ny^n$ in which every term is of degree n, is called a homogenous function of degree n in x and y.

Euler's theorem : If u is a homogenous function of degree 'n' in x and y then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

For any number of variables,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+.....+w\frac{\partial u}{\partial w}=nu$$

Example 8. If u be a homogenous function of degree n, then show that

(i) $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}$ (ii) $x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}$

(iii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$

Soln: Since u is a homogeneous function of degree n, hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \qquad \ldots (i)$$

(i) Differentiating both sides partialy with respect to x, we get

$$x\frac{\partial^2 u}{\partial x^2}+1.\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x} \Rightarrow x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x} \qquad \ldots (ii)$$

(ii) Again differentiating (i) partially w.r.t. y

$$x\frac{\partial u}{\partial y\partial x}+y\frac{\partial u}{\partial y^2}+1.\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y} \Rightarrow x\frac{\partial^2 u}{\partial x\,\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y} \qquad \ldots (iii)$$

(iii) Mutiplying (ii) by x and (iii) by y and adding,

$$x\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=n(n-1)u,$$

Example 9. If $u=\tan^{-1}\frac{x^3+y^3}{x-y}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

Soln: We have $u=\tan^{-1}\frac{x^3+y^3}{x-y}$, *i.e.*, $\tan u=\frac{x^3+y^3}{x-y}$

Let $z=\tan u=\frac{x^3+y^3}{x-y}$

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Hence z is a homogeneous function of degree (3-1)=2; $\Rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$

$$\Rightarrow x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=2\tan u \Rightarrow x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\tan u}{\sec^2 u}=2\sin u\cos u=\sin 2u$$

Note:

(1) If $u=f(x_1,x_2,x_3,.....x_n)$ and $x_1,x_2,x_3,.....x_n$ are all functions of t, then

$$\frac{du}{dt}=\frac{\partial u}{\partial x_1}.\frac{dx_1}{dt}+\frac{\partial u}{\partial x_2}.\frac{dx_2}{dt}+\frac{\partial u}{\partial x^3}.\frac{dx_3}{dt}+....+\frac{\partial u}{\partial x_n}.\frac{dx_n}{dt}$$

(2) If $u = f(x_1, x_2, x_3, ...x_n)$ and $x_2, x_3, ...x_n$ are all functions of x_1, then

$$\frac{du}{dx_1} = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}.\frac{dx_2}{dx_1} + ... + \frac{\partial u}{\partial x_n}.\frac{dx_n}{dx_1}$$

(3) If $u = f(x_1, x_2, x_3, ...x_n)$, then

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \frac{\partial u}{\partial x_3}dx_3 + ... \frac{\partial u}{\partial x_n}dx_n$$

(4) If x and y are related by the equation $f(x, y) = \text{constant}$, then

$$\frac{dy}{dx} = -\frac{(\partial f / \partial x)}{(\partial f / \partial y)}$$

Example 10. Find $\frac{dz}{dt}$ when $z = xy^2 + x^2y, x = at^2, y = 2at$

Soln: We have, $z = xy^2 + x^2y \Rightarrow \frac{\partial z}{\partial x} = y^2 + 2xy$ and $\frac{\partial z}{\partial y} = 2xy + x^2$

Also, $x = at^2, y = 2at \Rightarrow \frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$

Hence, $\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt} = (y^2 + 2xy)(2at) + (2xy + x^2)2a$

$= (4a^2t^2 + 2at^2.2at)2at + (2at^2.2at + a^2t^4)2a$

$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^4t^3 = 16a^3t^3 + 8a^3t^4 + 2a^4t^3$

Example 11. If $u = 4x^2 - y^2, x = e^t \cos t, y = 2e^t \sin t$, find $\frac{du}{dt}$

Soln: We have, $u = 4x^2 - y^2 \Rightarrow \frac{\partial u}{\partial x} = 8x$ and $\frac{\partial u}{\partial y} = -2y$

Also $x = e^t \cot t, \quad y = 2e^t \sin t$

Hence, $\frac{dx}{dt} = e^t \cos t - e^t \cdot \sin t, \frac{dy}{dt} = 2e^t \sin t + 2e^t \cos t$

Therefore, $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} = 8x(e^t \cos t - \sin t\, e^t) + (-2y).2e^t(\sin t + \cos t)$

$= 8e^t \cos t(e^t \cos t - \sin t\, e^t) - 4e^t(2e^t \sin t)(\sin t + \cos t) = 8e^{2t}(\cos 2t - \sin 2t)$

Example 12. Find $\frac{du}{dx}$ if $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$

Soln: We have, $u = \sin(x^2 + y^2) \Rightarrow \frac{\partial u}{\partial x} = \cos(x^2 + y^2).2x$ and $\frac{\partial u}{\partial y} = \cos(x^2 + y^2).2y$

Given: $a^2x^2+b^2y^2=c^2 \Rightarrow 2a^2x+2b^2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{a^2x}{b^2y}$

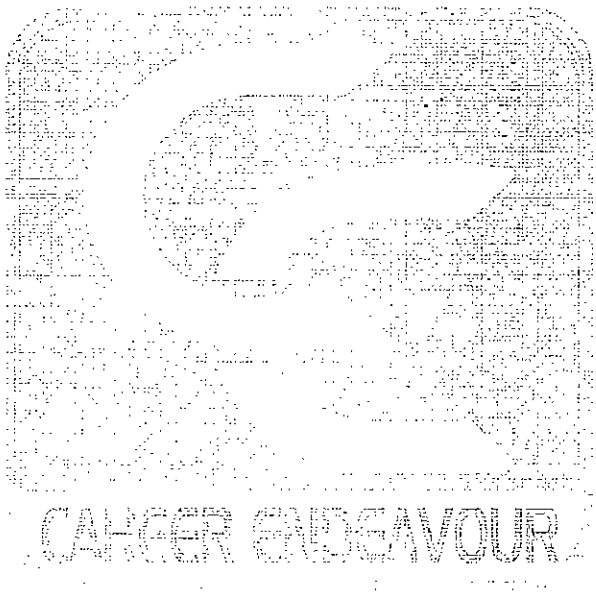
$$\Rightarrow \frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx}=2x\cos\left(x^2+y^2\right)+2y\cos\left(x^2+y^2\right)\left(-\frac{a^2x}{b^2y}\right)=2x\cos\left(x^2+y^2\right)\left(1-\frac{a^2x}{b^2y}\right)$$

Example 13. If $x^y+y^x=c$, find the value of $\frac{dy}{dx}$

Soln: Let $f(x,y)=x^y+y^x$

Then, $\frac{\partial f}{\partial x}=yx^{y-1}+y^x\log_e y$; Similarly, $\frac{\partial f}{\partial y}=x^y\log_e x+xy^{x-1}$

$$\therefore \quad \frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{y^x\log_e y+y^{y-1}}{x^y\log_e x+x\,y^{x-1}}$$



Chapter 11

JACOBIAN

Definition: If $u_1, u_2, u_3, \ldots\ldots, u_n$ be the functions of n variables $x_1, x_2, x_3, \ldots\ldots, x_n$ then the Jacobian of $u_1, u_2, \ldots\ldots u_n$ with respect to $x_1, x_2, x_3, \ldots\ldots x_n$ is

$$\frac{\partial(u_1, u_2, u_3, \ldots\ldots, u_n)}{\partial(x_1, x_2, x_3, \ldots\ldots, x_n)} = J(u_1, u_2, u_3, \ldots\ldots u_n) = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} & \cdots & \frac{\partial u_3}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Example 1. If $x = r\cos\theta, y = r\sin\theta$, then find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$

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Soln: $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$

Again we have, $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2}.2x = \cos\theta$; $\frac{\partial r}{\partial y} = \frac{1}{2}(x^2 + y^2)^{-1/2}.2y = \sin\theta$;

$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}}.\left(-\frac{y}{x^2}\right) = -\frac{\sin\theta}{r}$; $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$

$\therefore \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix} = 1/r(\cos^2\theta + \sin^2\theta) = 1/r$

Example 2. If $u_1 = \frac{x_2x_3}{x_1}$, $u_2 = \frac{x_3x_1}{x_2}$ and $u_3 = \frac{x_1x_2}{x_3}$, then prove that, $J(u_1,u_2,u_3) = 4$

Soln: We have $J(u_1,u_2,u_3) = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3^2} \end{vmatrix}$

$$= \frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & x_1x_2 \end{vmatrix} = \frac{(x_2x_3)(x_3x_1)(x_1x_2)}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4$$

Example 3. *If* $y_1 = (1-x_1)$, $y_2 = x_1(1-x_2)$, $y_3 = x_1x_2(1-x_3)$, ...

$y_n = x_1x_2...x_{n-1}(1-x_n)$, then prove that $\frac{\partial(y_1,y_2,....,y_n)}{\partial(x_1,x_2,.....x_n)} = (-1)^n x_1^{n-1}x_2^{n-2}.....x_{n-1}$

Soln: $\frac{\partial(y_1,y_2,....,y_n)}{\partial(x_1,x_2,.....,x_n)} = \frac{\partial y_1}{\partial x_1}\cdot\frac{\partial y_2}{\partial x_2}\cdot\cdots\cdot\frac{\partial y_n}{\partial x_n} = (-1).(-x_1).(-x_1x_2).....(-x_1x_2.....x_{n-1})$

$= (-1)^n x_1^{n-1}x_2^{n-2}.....x_{n-1}$

Jacobian of Implicit functions:

If $u_1,u_2,u_3,.....u_n$ and $x_1,x_2,.....x_n$ are implicitly connected by n equations as:

$f_1(u_1,u_2,.....u_n,x_1,x_2,.....,x_n) = 0$

$f_2(u_1,u_2,....,u_n,x_1,x_2,.....x_n) = 0$

.....

.....

$f_n(u_1,u_2,....,u_n,x_1,x_2,.....,x_n) = 0$

then $\frac{\partial(f_1,f_2,.....f_n)}{\partial(u_1,u_2,....u_n)}\cdot\frac{\partial(u_1,u_2,.....,u_n)}{\partial(x_1,x_2,....,x_n)} = (-1)^n\frac{\partial(f_1,f_2,....,f_n)}{\partial(x_1,x_2,....x_n)}$

Note : The above result is a generalised result of the elementary theorem $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$

Example 4. If $x+y+z = u$, $y+z = uv$, $z = u\,v\,w$, find the value of the Jacobian of x, y, z with respect to u, v, w.

Soln: Let, $f_1 = x+y+z-u = 0$

$f_2 = y+z-uv = 0$

$f_3 = z-uvw = 0$

Then, $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} \div \dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$

$$= -\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} \div \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -(-1)(-u)(-uv) \div (1)(1)(1) = u^2v$$

Jacobian of function of function:

Theorem : If $u_1, u_2, u_3, \ldots\ldots u_n$ are functions of $y_1, y_2, y_3, \ldots, y_n$ and $y_1, y_2, y_3, \ldots, y_n$ are functions of $x_1, x_2, x_3, \ldots\ldots x_n$ then,

$$\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,\ldots\ldots,x_n)} = \frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(y_1,y_2,\ldots,y_n)} \cdot \frac{\partial(y_1,y_2,\ldots,y_n)}{\partial(x_1,x_2,\ldots\ldots,x_n)}$$

Example 5. If $u^3+v^3+w^3 = x+y+z, u^2+v^2+w^2 = x^3+y^3+z^3$, $u+v+w = x^2+y^2+z^2$, then show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}$$

Soln: Let, $f_1 \equiv u^3+v^3+w^3-x-y-z = 0$, $f_2 \equiv u^2+v^2+w^2-x^3-y^3-z^3 = 0$

and $f_3 \equiv u+v+w-x^2-y^2-z^2 = 0$

We have, $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \cdot \dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} \div \dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$

$$= -\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} \div \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = -\begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix} \div \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2b & 2w \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} \div 6\begin{vmatrix} u^2 & v^2 & w^2 \\ u & v & w \\ 1 & 1 & 1 \end{vmatrix} = \frac{(y-z)(z-x)(x-y)}{(u-v)(u-w)(w-u)}$$

Theorem: Let $u_1, u_2, ..., u_n$ be functions of n independent variables $x_1, x_2, ...x_n$. The necessary and sufficient condition that the function be connected by a relation $f(u_1, u_2, ..., u_n) = 0$ is that the Jacobian $\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)}$ vanishes identically.

Example 6. Show that the functions: $u = 3x + 2y - z$, $v = x - 2y + z$ and $w = x(x + 2y - z)$ are not independent, and find the relation between them.

Soln: If the given functions u, v, w are not independent, then we must have $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ equal to zero, or

$$J = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -1 \\ (2x+2y-z) & 2x & -x \end{vmatrix} = 0$$

Adding R_2 to R_1, we get

Since, $J = 0$, hence the functions are not independent. Now we have to find the relation between u, v, w.

Clearly, $u + v = 4x$ and $u - v = 2(x + 2y - z)$

$$\therefore \quad (u+v).(u-v) = 8x(x+2y-z) = 8w \Rightarrow u^2 - v^2 = 8w$$

Example 7. If $u = \frac{x+y}{z}, v = \frac{y+z}{x}, w = \frac{y(x+y+z)}{xz}$, show that u, v, w are not independent and find the relation between them.

Soln: We have

$$J = \begin{vmatrix} \frac{1}{z} & \frac{1}{z} & -\frac{(x+y)}{z^2} \\ -\frac{(y+z)}{x^2} & \frac{1}{x} & \frac{1}{x} \\ -\frac{y^2-yz}{x^2z} & \frac{x+2y+z}{xz} & \frac{-xy-y^2}{xz^2} \end{vmatrix}$$

taking, $\frac{1}{z^2}, \frac{1}{x^2}$ and $\frac{1}{x^2z^2}$ common from R_1, R_2 and R_3 respectively, we get

$$J = \frac{1}{x^4z^4} \begin{vmatrix} z & z & -(x+y) \\ -(y+z) & x & x \\ -(y^2z+yz^2) & \frac{x+2y+z}{xz} & \frac{-xy-y^2}{xz^2} \end{vmatrix}$$

taking $\frac{1}{z^2}, \frac{1}{x^2}$ and $\frac{1}{x^2z^2}$ common from R_1, R_2 and R_3 respectively, we get

$$J = \frac{1}{x^4z^4} \begin{vmatrix} z & z & -(x+y) \\ -(y+z) & x & x \\ -(y^2z+yz^2) & x^2z+2xyz+xz^2 & -(x^2y+xy^2) \end{vmatrix} z$$

Applying $C_2 - C_1$ and $C_3 - C_1$,

$$= \frac{1}{x^4 z^4}\begin{vmatrix} z & 0 & -(x+y+z) \\ -(y+z) & (x+y+z) & (x+y+z) \\ -yz(y+z) & z(x^2+y^2+2xy)+z^2(x+y) & -xy(x+y)+yz(y+z) \end{vmatrix}$$

$$= \frac{1}{x^4 z^4}\begin{vmatrix} z & 0 & -(x+y+z) \\ -(y+z) & (x+y+z) & (x+y+z) \\ -yz(y+z) & z(x+y)(z+y+z) & (x+y+z)(yz-xy) \end{vmatrix}$$

$$= \frac{(x+y+z)^2}{x^4 y^4}\begin{vmatrix} z & 0 & -1 \\ -(y+z) & 1 & 1 \\ -yz(y+z) & z(x+y) & (yz-xy) \end{vmatrix}$$

Applying $C_1 + zC_3$, we get

$$= \frac{(x+y+z)^2}{x^4 z^4}(-1)\begin{vmatrix} -y & 1 \\ -yz(y+x) & z(x+y) \end{vmatrix} = \frac{(x+y+z)^2}{x^4 z^4}\left[-yz(x+y)+yz(x+y)\right] = 0$$

Since $J = 0$, hence the given functions are not independent.

Again. $uv = \dfrac{xy+y^2+yz+zx}{zx} = \dfrac{y(x+y+z)}{xz} + 1 = w + 1$

$\therefore$ $uv = w + 1$ is the required relation between them.

Example 8. Consider the integral $\iiint_V f(x, y, z)\, dV_1 = \iiint_V f(r, \theta, \phi)\, dV$, if $dV_1 = dx\, dy\, dz$ and $dV_2 = J\, dr\, d\theta\, d\phi$, then the value of J is

(a) $r \sin\theta$ (b) $r^2 \sin\theta$ (c) $r \sin\phi$ (d) $r^2 \sin\phi$

Soln: Here, J is the Jacobian from Cartesian (x, y, z) to spherical polar (r, θ, ϕ)

We have, $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial r & \partial r/\partial\theta & \partial r/\partial\phi \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}$$

On solving, this gives $= r^2 \sin\theta$. Therefore, $J = r^2 \sin\theta$.

Correct option is (b)

Example 9. If $z_1 = \dfrac{\omega_2\,\omega_3}{\omega_1}$, $z_2 = \dfrac{\omega_1\,\omega_3}{\omega_2}$, $z_3 = \dfrac{\omega_1\,\omega_2}{\omega_3}$, then the Jacobian (J) from $(z_1\ z_2\ z_3)$ to $(\omega_1\ \omega_2\ \omega_3)$ is equal to

(a) 3 (b) 2 (c) 4 (d) 5

Soln: $$J=\frac{\partial(z_1\ z_2\ z_3)}{\partial(\omega_1\ \omega_2\ \omega_3)}=\begin{vmatrix}\partial z_1/\partial\omega_1 & \partial z_1/\partial\omega_2 & \partial z_1/\partial\omega_3\\ \partial z_2/\partial\omega_1 & \partial z_2/\partial\omega_2 & \partial z_2/\partial\omega_3\\ \partial z_3/\partial\omega_1 & \partial z_3/\partial\omega_2 & \partial z_3/\partial\omega_3\end{vmatrix}$$

$$=\begin{vmatrix}-\omega_2\omega_3/\omega_1^2 & \omega_3/\omega_1 & \omega_2/\omega_1\\ \omega_3/\omega_2 & -\omega_1\omega_3/\omega_2^2 & \omega_1/\omega_2\\ \omega_2/\omega_3 & \omega_1/\omega_3 & -\omega_1\omega_2/\omega_3^2\end{vmatrix}$$

$$=\frac{1}{\omega_1^2\ \omega_2^2\ \omega_3^2}\begin{vmatrix}-\omega_2\omega_3 & \omega_1\omega_3 & \omega_1\omega_2\\ \omega_2\omega_3 & -\omega_1\omega_3 & \omega_1\omega_2\\ \omega_2\omega_3 & \omega_1\omega_3 & -\omega_1\omega_2\end{vmatrix}$$

$$=\frac{(\omega_2\omega_3)(\omega_1\omega_3)(\omega_1\omega_2)}{\omega_1^2\ \omega_2^2\ \omega_3^2}\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=4$$

Therefore, $J=4$.

Correct option is (c)

Example 10. If $u=x^2-y^2$, $v=2xy$, $x=r\cos\theta$, $y=r\sin\theta$, then the value of Jacobian $J=\dfrac{\partial(u,v)}{\partial(r,\theta)}$ is

(a) $\frac{1}{2}\left(x^2+y^2\right)^{3/2}$ (b) $2\left(x^2+y^2\right)^{1/2}$ (c) $4\left(x^2+y^2\right)^{3/2}$ (d) $4\left(x^2+y^2\right)^{1/2}$

Soln: We have $J=\dfrac{\partial(u,v)}{\partial(r,\theta)}=\dfrac{\partial(u,v)}{\partial(x,y)}\times\dfrac{\partial(x,y)}{\partial(r,\theta)}$

Here, $\dfrac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{vmatrix}=\begin{vmatrix}2x & -2y\\ 2y & 2x\end{vmatrix}=4\left(x^2+y^2\right)$

and $\dfrac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\partial x/\partial r & \partial x/\partial\theta\\ \partial y/\partial r & \partial y/\partial\theta\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$

Therefore, $J=4\left(x^2+y^2\right)r=4\left(x^2+y^2\right)^{3/2}$.

Correct option is (c)

Example 11. If $u=xyz$, $v=x^2+y^2+z^2$, $w=x+y+z$, then the value of Jacobian, $J=\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ is

(a) $-2(x-y)(y-z)(z-x)$ (b) $-\dfrac{1}{2(x-y)(y-z)(z-x)}$

(c) $-\dfrac{(x-y)(y-z)(z-x)}{2}$ (d) None of these

Soln: $J=\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\partial u/\partial x & \partial u/\partial y & \partial u/\partial z\\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z\\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z\end{vmatrix}=\begin{vmatrix}yz & xz & xy\\ 2x & 2y & 2z\\ 1 & 1 & 1\end{vmatrix}$

$$=\begin{vmatrix}z(y-x) & x(z-y) & xy\\ -2(y-x) & -2(z-y) & 2z\\ 0 & 0 & 1\end{vmatrix}\quad[C_1\to C_1-C_2 \text{ and } C_2\to C_2-C_3]$$

$$=(y-x)(z-y)\begin{vmatrix}z & x & xy\\ -2 & -2 & 2z\\ 0 & 0 & 1\end{vmatrix}$$

$$=(x-y)(y-z)\{-2z+2x\}$$

$$=-2(x-y)(y-z)(z-x)$$

Therefore, $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=-\dfrac{1}{2(x-y)(y-z)(z-x)}$.

Correct option is (b)

Example 12. Let x and y are Cartesian variable which transformed to another variable u and v such that $x=2u+3v$ and $y=2u-3v$, then the value of $|J|$ is _____. ($|J|$ denotes the modulus of J).

Soln: $J=\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{vmatrix}=\begin{vmatrix}2 & 3\\ 2 & -3\end{vmatrix}=-6-6=-12$

Hence, $|J|=12$.

Correct answer is (12)

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Chapter 12

DIRAC DELTA , BETA & GAMMA FUNCTION

Dirac Delta function:

$$\delta(x-a)=0 \quad x\neq a$$
$$=\infty \quad x=a \quad (\text{very high})$$

Such that $\int_{-\infty}^{\infty}\delta(x-a)\,dx=1$ i.e. area under the curve should be equal to unity.

Note: If x has a dimension of length, then Dirac delta function will have a dimension of $(\text{length})^{-1}$.

Some representations of Dirac Delta function:

(1) $\lim_{\sigma\to 0} R_\sigma(x)=\delta(x-a)$ where rectangular function $R_\sigma(x)=\dfrac{1}{2\sigma}$ $a-\sigma<x<a+\sigma$

$=0$ otherwise

(2) $\lim_{\sigma\to 0} G_\sigma(x)=\delta(x-\mu)$ where Gaussian function $G_\sigma(x)=\dfrac{1}{\sqrt{2\pi}\,\sigma}\exp\left[\dfrac{-(x-\mu)^2}{2\sigma^2}\right]$

(3) $\lim_{\varepsilon\to 0} L_\sigma(x)=\delta(x-x_0)$ where Lorentzian function $L_\sigma(x)=\dfrac{1}{\pi}\dfrac{\varepsilon}{(x-x_0)^2+\varepsilon^2}$

(4) Integral representation: $\dfrac{1}{2\pi}\int_{-\infty}^{\infty}e^{\pm ik(x-x_0)}dk=\delta(x-x_0)$

Some properties of Dirac Delta function:

(1) $\int_{-\infty}^{\infty}f(x)\delta(x-a)dx=f(a)$

(2) $\int_{-\infty}^{\infty}f'(x)\delta(x-a)dx=-f'(a)$

(3) $\delta(x-a)=\delta(x-a)$

(4) $\delta[c(x-a)]=\dfrac{1}{|c|}\delta(x-a)$ (c is constant)

(5) $\delta(x^2-a^2)=\dfrac{1}{|2a|}[\delta(x-a)+\delta(x+a)]$

(6) $\int_{-\infty}^{\infty}\delta(x-a)\delta(x-b)dx=\delta(a-b)$

(7) $F[\delta(x-a)]=\frac{1}{\sqrt{2\pi}}e^{isa}$

(8) $L[\delta(x-a)]=e^{-sa}$

Example 1: Evaluate the following integrals

Soln. (i) $\int_0^{\pi}\cos x\delta(x-\pi)\,dx=\cos\pi=-1$

(ii) $\int_{-2}^{2}x^3\delta(x+1)\,dx=(-1)^3=-1$

(iii) $\int_{-2}^{2}(2x+3)\delta(3x)\,dx=\frac{1}{3}\int_{-2}^{2}(2x+3)\delta(x)\,dx=1$

(iv) $\int_0^2(x^3+3x+2)\delta(x-1)dx=2$

(v) $\int_{-\infty}^{\infty}\cos x\delta(x^2-\pi^2)dx=\int_{-\infty}^{\infty}\cos x\frac{1}{|2\pi|}[\delta(x=\pi)+\delta(x+\pi)]dx=\frac{1}{2\pi}[\cos\pi+\cos(-\pi)]=-\frac{1}{\pi}$

Representation of Dirac delta function in 3-D:

(1) Cartesian Co-ordinates: $\delta^3(\vec{r}-\vec{r}_0)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$

(2) Spherical polar Co-ordinates: $\delta^3(\vec{r}-\vec{r}_0)=\frac{1}{r^2\sin\theta}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\varphi-\varphi_0)$

(3) Cylindrical Co-ordinates: $\delta^3(\vec{r}-\vec{r}_0)=\frac{1}{\rho}\delta(\rho-\rho_0)\delta(\varphi-\varphi_0)\delta(z-z_0)$

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Gamma Function: $\int_0^{\infty}e^{-x}x^{n-1}dx=\Gamma n$

Some important relations:

(1) $\Gamma(n+1)=n\Gamma n=n!$ (2) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

(3) Gamma of zero and negative integers is not defined.

Example 2: Evaluate the following integral $\int_0^{\infty}x^2\,e^{-2x^2}dx$

Soln. $\int_0^{\infty}x^2\,e^{-2x^2}dx$, assuming $2x^2=t\Rightarrow 4xdx=dt\quad\Rightarrow dx=\frac{dt}{4x}=\frac{1}{2\sqrt{2t}}dt$, \

we get $I=\frac{1}{4\sqrt{2}}\int_0^{\infty}e^{-t}t^{1/2}dt=\frac{1}{4\sqrt{2}}\Gamma\left(\frac{3}{2}\right)=\frac{1}{8\sqrt{2}}\sqrt{\pi}$

Beta function:

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)$$

Some important relations:

(1) $\beta(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$ (2) $\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \dfrac{1}{2}\beta\left(\dfrac{p+1}{2},\dfrac{q+1}{2}\right) = \dfrac{1}{2}\dfrac{\Gamma\left(\dfrac{p+1}{2}\right).\Gamma\left(\dfrac{q+1}{2}\right)}{\Gamma\left(\dfrac{p+q+2}{2}\right)}$

(where p and q are positive integers of fractions)

(3) $\Gamma x.\Gamma(1-x) = \dfrac{\pi}{\sin\pi x}$

Example 3: Evaluate the integral $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$

Soln. $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta = \dfrac{1}{2}\dfrac{\Gamma\left(\dfrac{3}{4}\right).\Gamma\left(\dfrac{1}{4}\right)}{\Gamma(1)} = \dfrac{1}{2}\dfrac{\pi}{\sin\dfrac{\pi}{4}} = \dfrac{\pi}{\sqrt{2}}$

Example 4: The value of $\int_0^3 t^2\delta(3t-6)\,dt$ is_____ (upto one decimal place)

Soln. $\int_0^3 t^2\delta(3t-6)\,dt \int_0^3 t^2\delta\left[3(t-2)\right]dt = \dfrac{1}{3}\int_0^3 t^2\delta(t-2)\,dt = \dfrac{4}{3} = 1.3$

Example 5: The value of the integral $\int_0^\infty dx\,x^9\exp(-x^2)$ is **[TIFR 2013]**

(a) 20160 (b) 12 (c) 18 (d) 24

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Soln. $\int_0^\infty dx\,x^9 e^{-x^2}$ [put $x^2 = z \Rightarrow 2xdx = dz$]

$= \int_0^\infty e^{-z}\cdot z^4\cdot\dfrac{dz}{2}$

$= \dfrac{1}{2}\int_0^\infty e^{-z}\cdot z^4\cdot dz$ [Using Gamma integral]

$= \dfrac{1}{2}\Gamma 5 = 12$ $\int_0^\infty e^{-p}\cdot p^{n-1}dp = \Gamma n$

Correct option is (b)

Example 6: The integral $\int_{-\infty}^{\infty} dx\,\delta(x^2-\pi^2)\cos x$, evaluates to **[TIFR 2013]**

(a) -1 (b) 0 (c) $\frac{1}{\pi}$ (d) $-\frac{1}{\pi}$

Soln. $\int_{-\infty}^{\infty} dx\delta\left(x^2-\pi^2\right)\cos x dx$

Applying the property, $\delta\left(x^2-a^2\right)=\frac{1}{|2a|}\left[\delta\left(x+a\right)+\delta\left(x-a\right)\right]$

$=\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\left[\delta\left(x+\pi\right)+\delta\left(x-\pi\right)\right]\cos x dx$

Applying the property, $\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx=f(a)$

$=\frac{1}{2\pi}\left[\cos\pi+\cos(-\pi)\right]=-\frac{1}{\pi}$

Correct option is (d)

Example 7: The Dirac delta function $\delta(x)$ satisfies the relation $\int_{-\infty}^{\infty} f(x)\delta(x)dx=f(0)$ for a well behaved function $f(x)$. If x has the dimension of momentum then **[JEST 2014]**

(a) $\delta(x)$ has the dimension of momentum (b) $\delta(x)$ has the dimension of (mometum)^2

(c) $\delta(x)$ is dimensionless (d) $\delta(x)$ has the dimension of (momentum)$^{-1}$.

Soln. $\int_{-\infty}^{\infty} f(x)\delta(x)dx=f(0)$

Since, $f(x)$ and $f(0)$ have same dimension, then $\delta(x)dx$ should be dimensionless.

If x has dimension of momentum, then $\delta(x)$ has a dimension of (momentum)$^{-1}$

Correct option is (d)